OPERADS IN ALGEBRA, TOPOLOGY AND PHYSICS

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Part I

Chapter 1

Introduction and History

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Operads involve an abstraction of a family of composable functions of several variables $Map(X^n, X)$ together with an action of permutations of variables. As such, they have appeared in various ways over the last century and have recently had a great variety of applications which have provided the impetus for this book.

Operads as such were originally studied as a tool in homotopy theory, specifically for iterated loop spaces, but the theory of operads has recently received new inspirations from homological algebra, category theory, algebraic geometry and mathematical physics. The name *operad* and the formal definition appears first in the early 70's in J. Peter May's "The Geometry of Iterated Loop Spaces" [90], but a year or more earlier, Boardman and Vogt described the same concept under the name 'categories of operators in standard form,' inspired by PROPs and PACTs of Adams and Mac Lane. In fact, there is an abundance of pre-history. Weibel [133] points out that the concept first arose a century ago in A.N. Whitehead's "A Treatise on Universal Algebra" [135], published in 1898.

Whitehead was actually describing quadratic operads in the category of vector spaces, specifically those generated by a single binary product. In sections 20-22, under the title 'complete algebraic system,' he fixes a product and refers to the order of higher products determined by iterated compositions thereof. In section 92 (p. 172) he defines 'invariant equations of condition;' in modern language these are exactly the relations on the free operad on a single binary operation needed to produce the operad in question. He goes on in section 93 to add associativity explicitly and studies the resulting theory of symmetric and exterior algebras.

Operads (or, more precisely, 'systems with compositions') arise again in the mid 1950's in the work of Lazard on formal groups.

1.1. Lazard's formal group laws.

In 1954, Michel Lazard, studying formal group laws, introduced the structure of 'analyseur' to axiomatize composition laws. This structure is a refinement of a more basic structure, 'a family of polynomial functions,' which is probably the earliest explicit example of an operad in the category of modules over a ring. Of course the Hochschild cochain complex, which had been introduced a decade earlier, has an operadic structure, but this was not really explicit until the work of Gerstenhaber [31] some nine years later. The description we give here follows Lazard's review of his work in Séminaire Bourbaki 1954-1955 [70] and Chevalley's extensive review [15] of Lazard's full length paper [69].

Let $\mathbf{k}$ be a commutative ring with unity and E, a unitary $\mathbf{k}$-module. A not necessarily linear function $f \in \mathrm{Map}(E^{\oplus n}, E)$ will be called a function of n variables and will be denoted $f(x_1, \dots, x_n)$. The set of such functions has the structure of a $\mathbf{k}$-module. An argument x_i of $f(x_1, \dots, x_n)$ is called neutral if $f(a_1, \dots, a_n) = f(b_1, \dots, b_n)$ whenever $a_j = b_j$ for all $j \neq i$; the function is then said to be neutral in that argument. Composition of a function of n arguments with any one of the natural projections $E^{\oplus(n+k)} \to E^{\oplus n}$ 'adds neutral arguments,' that is, defines a function of $n+k$ arguments which does not depend on k of the arguments. Similarly we can reduce a function by deleting its neutral arguments; i.e., by factoring it through the projection that eliminates those arguments. There are then several conditions that describe what Lazard calls a *family* $\mathcal{P}$ of polynomial functions on a $\mathbf{k}$-module E.

A family is a graded $\mathbf{k}$-module,

$$\mathcal{P} = \bigoplus_{0<n} \mathcal{P}^n, \text{ where } \mathcal{P}^n \subset Map(E^{\oplus n}, E),$$

which is closed under composition of one function f of m variables with m 'input' functions g_i, $1 \leq i \leq n$, each depending on the same number n of arguments. We also assume that $\mathcal{P}$ contains the identity function $id_E : E \to E$.

The next step is to introduce the concept of homomorphism and equivalence, so that one can eventually define an abstract object independent of the particular $\mathbf{k}$-module which appears as the domain of definition of the functions.

A *homomorphism* $\phi : \mathcal{P} \to \mathcal{P}'$ from a family $\mathcal{P}$ of polynomial functions on a module E to a family $\mathcal{P}'$ of polynomial functions on a module E' respects the number of variables, composition and the corresponding identity maps. An isomorphism of families is then an invertible homomorphism.

An *'analyseur incomplet'* is defined as an equivalence class of families of polynomial functions, under the equivalence relation given by isomorphism. A (complete) *analyseur* is defined by a completion process analogous to the completion of a polynomial ring to a formal power series ring.

A basic example of analyseur is given by the polynomial functions without constant term on an infinite field. A *group law* in an analyseur $\mathcal{A}$ is an element of $\mathcal{A}^2$ satisfying

$$f(x,y) = x + y \quad \text{modulo terms of homogeneity} \geq 2, \text{ and}$$
$$f(f(x,y),z) - f(x,f(y,z)) = 0.$$

Lazard defines a q-germ of a group law as one that satisfies the associativity axiom modulo terms of homogeneity $q+1$. He then introduces a cohomological obstruction to the extension of a q-germ to a $q+1$ germ which anticipates the Gerstenhaber deformation theory of algebras, see [32].

1.2. PROPs and PACTs.

Towards the end of the 1950's, Adams and Mac Lane were motivated by problems in algebraic topology, namely situations such as coproducts in iterated bar constructions or simplicial chain complexes which were commutative only 'up to higher homotopies.' There are, to quote Mac Lane [76], 'so many of them that general conceptual methods are needed for their treatment.' They therefore invented PROPs and PACTs, although the first publication from this collaboration did not appear in 1963 [75].

PROPs were categories $\mathbf{H}$ (an abbreviation of PROduct and Permutation category) with objects the natural numbers $\{0, 1, 2, ...\}$ and 'Hom-sets' of maps from m to n denoted

$$\mathbf{H}\binom{m}{n}.$$

There is a strictly associative 'monoidal structure' (denoted by $\otimes$) given by composition on 'Hom-sets'

$$\otimes : \mathbf{H}\binom{m}{n} \times \mathbf{H}\binom{n}{p} \to \mathbf{H}\binom{m}{p} \tag{1.1}$$

which is suitably symmetric. In particular, $\mathbf{H}\binom{n}{n}$ contains the symmetric group Σ_n as a specified subgroup; thus the 'tensor product' (1.1) induces a left Σ_m-, right Σ_n-action on $\mathbf{H}\binom{m}{n}$.

Let us point out that the convention given here is opposite to the convention we use in the book and follows the 'categorial' composition rule for arrows, not for maps. Intuitively, the space $\mathbf{H}\binom{m}{n}$ parameterizes operations with m inputs and n outputs.

PACTs were PROPs in which the Hom-sets $\mathbf{H}\binom{m}{n}$ were dg-modules over a commutative ring with the 'tensor' product (1.1) and symmetric group actions being compatible with the dg-structure.

Of special importance is the PROP of endomorphisms $\mathbf{End}_X\binom{m}{n}$ of maps $X^{\odot m} \to X^{\odot n}$ for any object X in a monoidal category with a product $\odot$. An *action* of a PROP $\mathbf{H}$ on an object X is then a map of PROPs $\mathbf{H} \to \mathbf{End}_X$. In this situation we also say that X is an *algebra over* $\mathbf{H}$ or that X is a $\mathbf{H}$-*space*.

1.3. Non-Σ-operads and operads.

In addition to the publication of PACTs and PROPs, 1963 was a good year for operads with the publication of Gerstenhaber 'comp algebras' in his study of Hochschild cohomology [31] and Stasheff's 'associahedra' for his homotopy characterization of loop spaces [117]. With hindsight, we can say that these constructions are examples of what are now often called 'non-Σ operads,' the 'non-Σ' indicating that we do not assume any symmetric group action.

In 1966/67, Mac Lane ran a seminar on PACTs and PROPs at the University of Chicago. Among the participants were Adams, Boardman, Stasheff, and Vogt [130]. Boardman's linear isometries PROP (later reincarnated as an operad) provided the key to developing an understanding of infinite loop spaces, as was his 'little cubes PROP' the key to understanding iterated loop spaces in general. The following year, May built upon Boardman and Vogt's structures to finalize the notion he dubbed 'operad.'

The model is the set $Map(X^n, X)$ of all functions of n variables in a set X for all $n \geq 1$. Let

$$\circ_i : Map(X^n, X) \times Map(X^m, X) \longrightarrow Map(X^{n+m-1}, X) \tag{1.2}$$

be given, for $1 \leq i \leq n$, by

$$(f \circ_i g)(x_1, \cdots, x_{m+n-1}) = f(x_1, \cdots, x_{i-1}, g(x_i, \cdots, x_{i+m-1}), x_{i+m}),$$

A *non-Σ operad* $(\mathcal{O}, \circ_i)$ in the category of sets consists of a sequence of sets $\{\mathcal{O}(n)\}_{n\geq 1}$, a unit map $1 \in \mathcal{O}(1)$ and products $\circ_i : \mathcal{O}(m) \times \mathcal{O}(n) \to \mathcal{O}(m+n-1)$ for $m, n \geq 1$ satisfying the relations manifest in the example $\mathcal{O}(n) := Map(X^n, X)$.

Analogous objects for the category of graded modules were introduced by Gerstenhaber [31] under the name *comp algebra*.

Non-Σ-operads can be defined in any monoidal category (apparently first remarked by Malraison in the mid 70s [77]) and operads in any symmetric monoidal category (see section **??**)[1]. May's original definition, motivated by the topological applications, was in the category of compactly generated topological spaces.

Passing to associated chain complexes and hence to the category of differential graded modules was implicit already in Stasheff's work, but apparently was not formalized until the early 1980s by Hinich and Schechtman [50].

An *operad* is a non-Σ-operad $(\mathcal{O}, \circ_i)$ with an action of the symmetric group Σ_j on $\mathcal{O}(j)$ for each j. These actions are compatible with the $\circ_i$ operations as manifest in the example of the *endomorphism operad* $\mathcal{E}nd_X := \{Map(X^n, X)\}_{n\geq 1}$.

Observe that each non-Σ operad can be made an operad replacing, for each $n \geq 1$, $\mathcal{O}(n)$ by $\mathcal{O}(n) \times \Sigma_n$ and extending the structure operations $\circ_i$ to be compatible with the action of

[1] references to future sections, theorems, definitions, etc., to be added

the symmetric groups. Thus there is no need to make any real distinction between operads and non-Σ operads.

May's original definition (see section **??**) includes $\mathcal{O}(0)$ which is a single point. More importantly, rather than deal with the individual $\circ_i$ operations, May's definition combines them into a single operation

$$\gamma : \mathcal{O}(k) \times \mathcal{O}(j_1) \times \cdots \times \mathcal{O}(j_k) \longrightarrow \mathcal{O}(j_1 + \cdots + j_k) \tag{1.3}$$

satisfying upon iteration the relations of associativity and symmetry manifest in the example of the endomorphism operad $\{Map(X^n, X)\}_{n\geq 1}$. In general, γ can be obtained, by iterating appropriate $\circ_i$'s, as

$$\gamma(f \times g_1 \times \cdots \times g_n) := (\cdots(((f \circ_1 g_1) \circ_{l_1+1} g_2) \circ_{l_1+l_2+1} g_3) \cdots) \cdots,$$

where l_i is the arity of g_i. In turn, using the identity $1 \in \mathcal{O}(1)$, the composition $f \circ_i g$ can be reconstructed as

$$f \circ_i g = \gamma(f \ \times 1 \times \cdots \times g \times \cdots \times 1) \text{ (g at the i-th position).}$$

Thus, under the presence of a unit, both approaches are equivalent. The advantage of the $\circ_i$-approach is that the defining operations are binary, thus giving rise to a natural grading of free objects (see also discussion in **??**). In our book, all operads will have a unit, or at least something very close to it, so we will understand both the γ- and $\circ_i$- formalisms as nothing more than two different ways of describing the same object.

Operads are important not in and of themselves but, like PROPs, through their representations, more commonly called 'algebras over operads.' An *algebra A over an operad $\mathcal{O}$* is a map of operads $\mathcal{O} \to \mathcal{E}nd_A$. This is just a fancy way of saying that an algebra A is a set (an object in the category) with a coherent system of maps $\mathcal{O}(n) \times A^n \to A$.

Each operad $\mathcal{O}$ generates a PROP $\mathbf{H}$ with $\mathbf{H}\binom{m}{1} := \mathcal{O}(m)$ for each $m \geq 1$ such that algebras over the PROP $\mathbf{H}$ are the same as algebras over the operad $\mathcal{O}$. But it is not true that all PROPs are generated by an operad. An example is provided by bialgebras (= Hopf algebras without antipode). They are objects consisting of a vector space V together with an associative multiplication $\mu : V \otimes V \to V$ and coassociative comultiplication $\Delta : V \to V \otimes V$ such that Δ is a morphism of algebras (equivalently, μ is a morphism of coalgebras).

While bialgebras are algebras over a certain PROP [80], operads cannot accommodate both the multiplication μ and the comultiplication Δ.

1.4. Theories.

Another formalization of the concept of a variety of algebras given by operations and laws without existential quantifiers was introduced in 1963 by Lawvere [68].

Let $\mathfrak{S}$ be the category whose objects are finite sets $[n] := \{1, \ldots, n\}$ and morphisms all set maps between these finite sets. The dual category $\mathfrak{S}^{\rm op}$ is a category with finite products in which every object $[m]$ is the n-fold product $[1]^{\times n}$ of the distinguished object $[1]$.

A (finitary) *algebraic theory* is then a category Θ with the same objects as $\mathfrak{S}^{\rm op}$, together with a faithful functor $\mathfrak{S}^{\rm op} \to \Theta$ that preserves objects and products. A Θ-*space*, or a Θ-*algebra* in a category $\mathcal{C}$ is a functor $A : \Theta \to \mathcal{C}$ preserving the products. The object $X := A([1]) \in \mathcal{C}$ is the *underlying space* of the algebra A.

The existence of the functor $\mathfrak{S}^{\rm op} \to \Theta$ means that each Θ-structure on a space X necessarily contains operations

$$\sigma^* : X^n \to X^m,\ \sigma^*(x_1, \ldots, x_n) = (x_{\sigma(1)}, \ldots, x_{\sigma(m)}),$$

for $\sigma : \{1, \ldots, m\} \to \{1, \ldots, n\} \in \mathfrak{S}([m], [n])$. In particular, each Θ-structure contains projections and permutations of variables.

Each operad $\mathcal{O}$ generates a theory Θ such that $\Theta([m], [1])$ is obtained from $\mathcal{O}(m)$ by adjoining all projections $[m] = [1]^{\times m} \to [1]$. Algebras in a category $\mathcal{C}$ over the theory Θ are the same as algebras over the operad $\mathcal{O}$ provided the monoidal structure of $\mathcal{C}$ is given by products. This construction was studied in detail by Boardman and Vogt in [10]; they call $\mathcal{O}$ the *spine* of the theory Θ.

On the other hand, it is not true that all theories are of this form. For example, Boolean rings where we require $x^2 = x$ cannot be described as algebras over an operad. Roughly speaking, a Lawvere theory is generated by an operad (spine) if and only if the defining axioms of the theory involve no repetitions and each axiom is homogeneous in the number of variables.

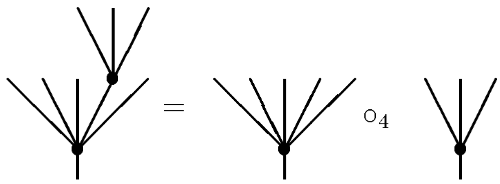
Since $\Theta([m], [n]) = \Theta([m], [1])^{\times n}$ for each $m, n \geq 1$, Lawvere's theories are intrinsically 'algebraic,' thus they, like operads, cannot accommodate objects with 'co-algebraic' operations, like coalgebras or Hopf algebras.

1.5. Tree operads.

In the early days, three examples were particularly important in the development of the subject; the linear isometries operad, the little cubes operad and the associahedra non-Σ operad (see respectively sections **??** and **??**). The underlying *tree operads* have received more attention as the subject has developed.

To quote Boardman and Vogt [10], 'the trees are inspired by the attempt to obtain a general composite operation from a collection of indecomposable operations'.

Let $\mathcal{PT}(n)$ be the set of planar trees with 1 root at the bottom and n leaves at the top, regarded as labeled from left to right: 1 through n. The collection $\mathcal{PT} = \{\mathcal{PT}(n)\}_{n\geq 1}$ forms a non-Σ-operad: given trees $f \in \mathcal{PT}(k), g \in \mathcal{PT}(j)$, for each $1 \leq i \leq k$, let $f \circ_i g$ be the tree obtained by grafting the root of g to the leaf of f labeled i.



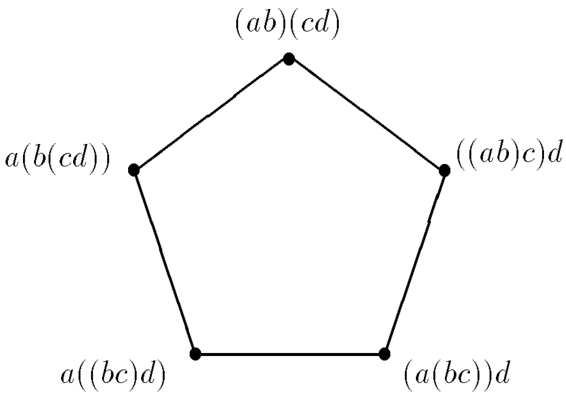
Similarly, let $\mathcal{T}(n)$ be the set of (non-planar) trees with 1 root and n leaves labeled (arbitrarily) 1 through n. The collection $\mathcal{T} = \{\mathcal{T}(n)\}_{n\geq 1}$ of tree spaces forms an operad by the same rule as for planar trees using the labeling: given trees $f \in \mathcal{T}(k), g \in \mathcal{T}(j)$, for each $1 \leq i \leq k$, let $f \circ_i g$ be the tree obtained by grafting the root of g to the leaf of f labeled i. The symmetric groups act by permuting the labels.

We will see later **??** that $\mathcal{PT}(n)$ and $\mathcal{T}(n)$ are respectively a free non-Σ-operad and a free operad and play a central role in many applications.

1.6. A_∞-spaces and loop spaces.

In 1957, Sugawara began to publish a series of papers [120, 121, 122] characterizing H-spaces and loop spaces. One of his most important results was a *recognition principle* in homotopy invariant terms identifying which H-spaces were of the homotopy type of a loop space. This led in 1961 to Stasheff's homotopy characterization of connected based loop spaces in terms of what we would now call the non-Σ-operad of associahedra [117].

In the course of simplifying Sugawara's recognition principle for loop spaces (by eliminating the use of homotopy inverses), Stasheff was led to consider higher homotopies for associativity, e.g. the pentagon



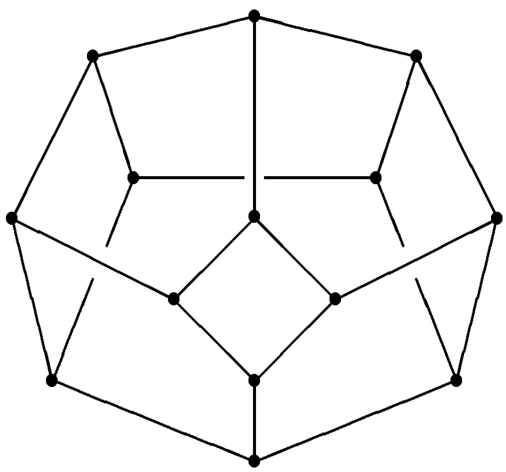


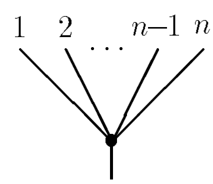
Figure 1.1: Saito's portrait of K_5.

where the sides represent a single application of a specific associating homotopy $h(a,b,c)$ from $a(bc)$ to $(ab)c$. For example, the bottom edge above from left to right is given by $h(a, bc, d)$.

While Stasheff continued this work on H-spaces as a student in Oxford, Frank Adams came to visit the Mathematical Institute and discussed his work with Mac Lane on PACTs and PROPs. With a key idea from Adams, Stasheff created the polytopes now known as *associahedra* [116, 117]. The name is due to Gil Kalai, a geometric combinatorialist [72, 71].

The associahedron K_n can be described now as a convex polytope with one vertex for each way of associating n ordered variables, that is, ways of inserting parentheses in a meaningful way in a word of n letters. For $n = 5$, a portrait due to Masahico Saito is in Figure 1.1.

To describe all the cells of K_n, the language of planar rooted trees is helpful, as Adams indicated to Stasheff. The dimension d-cells of K_n are all of the form $\prod_{i=1}^{k} K_{n_i}$, where $d = n_1 + \cdots + n_k - k$. In particular all the facets (cells of codimension 1) are of the form $K_r \times K_s$, $r + s = n + 1$. If we let the top dimensional cell of K_n be indexed by the *corolla*,



the planar tree with n leaves, then the facets of e.g. K_4 are labeled as in Figure 1.2.

In general, the facets are labeled by grafting the s-corolla to leaf i of the r-corolla. The corresponding inclusion of facets

$$\circ_i : K_r \times K_s \hookrightarrow K_{r+s-1}$$

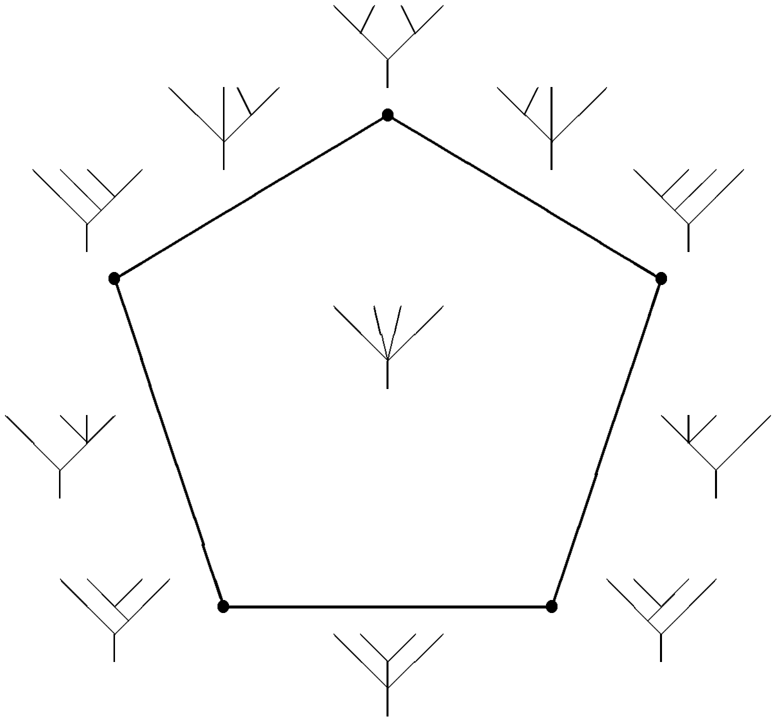


Figure 1.2: K_4 with facets labeled by trees.

makes the set of associahedra $\mathcal{K} = \{K_n\}_{n\geq 1}$ into a non-Σ-operad.

Jumping ahead in time, there was an 'open' problem in combinatorial geometry (posed around 1978 and solved in 1984 by Haiman) as to whether there was a *convex polytope* realizing the associahedron. As originally constructed, the K_n were convex but curvilinear, although linearized early on by Milnor (unpublished).

Thanks to Kapranov, who bridged between the topological and combinatorial communities, the combinatorialists realized that the solution predated the question. A particular realization as a truncation of the standard simplex and related to symplectic moment maps and toric varieties was presented by Shnider and Sternberg in their book ([109], second edition). The cover illustration describes K_5 as a truncated 3-simplex. The general case is described in more detail in an appendix to [115].

An *A_∞-space* Y is a topological space Y together with a family of maps

$$m_n : K_n \times Y^n \to Y$$

which fit together to make Y an algebra over the non-Σ-operad $\mathcal{K} = \{K_n\}_{n\geq 1}$. The main result is the following theorem.

Theorem. *A connected space Y (of the homotopy type of a CW-complex with a nondegenerate base point) has the homotopy type of a based loop space ΩX for some X if and only if Y admits the structure of an A_∞-space.*

This result, like Sugawara's, is the archetype of results in the theory of homotopy invariant structures on topological spaces. To show that a connected A_∞-space Y has the homotopy type of a loop space involves constructing a 'classifying space' BY, defined as a quotient

$$BY := \coprod_{n \geq 0} K_{n+2} \times Y^n / \sim$$

where the identification $\sim$ involves the structure maps m_n.

After his work categorizing loops spaces homotopy theoretically, Sugawara considered 'strongly homotopy commutative' strictly associative H-spaces as a step forward identifying two fold loop spaces $\Omega^2 X$. The loop multiplication on $\Omega^2 X$ must be homotopy commutative (cf. the standard proof that $\pi_1(\Omega X)$ is abelian), but the commuting homotopy must satisfy a whole family of coherence relations with regard to products of an arbitrary number of elements. In fact, for an associative H-space Y, the existence of an H-space structure $BY \times BY \to BY$ requires a coherent family of higher homotopies $h_n : I^{n-1} \times Y^n \to Y$ of which h_2 is a commuting homotopy and h_3 is related to Mac Lane's hexagon condition for coherence of a monoidal category.

Such a family $\{h_n\}$ is a special case of the A_∞-maps between A_∞-spaces X and Y which correspond precisely to maps $BX \to BY$ (see **??**). Sugawara's strong homotopy commutativity was a homotopy invariant description for a connected associative H-space having the homotopy type of the loop space on an H-space; such a characterization of a double loop space (the loop space of a loop space) awaited the invention of operads.

1.7. E_∞-spaces and iterated loop spaces.

A major force in the development of this sort of higher homotopy structure was the study of infinite loop spaces, originating with Bott periodicity which showed that the classifying space BU of the unitary K-theory had the homotopy type of the connected component of the double loop space $\Omega^2 BU$ and hence can be regarded as an iterated loop space $\Omega^n X_n$ for all n.

A space X_0 is called an *infinite loop space* if there is a sequence of spaces X_n and homotopy equivalences $X_n \sim \Omega X_{n+1}$ for $n \geq 0$.

The early work of Boardman and Vogt on this problem dealt with characterizing such spaces as 'homotopy everything' spaces or what are now called E_∞-spaces. Homotopy everything is a bit of an exaggeration (as they acknowledge in [10]) since an E_∞-operad action implies that, up to homotopy, we are dealing with a single binary operation.

In May's terminology [90], an operad $\mathcal{C} = \{\mathcal{C}(n)\}_{n \geq 1}$ is an E_∞-*operad* if and only if each $\mathcal{C}(n)$ is Σ_n-free and contractible. Similarly, a non-Σ-operad $\mathcal{C} = \{\mathcal{C}(n)\}_{n \geq 1}$ is an A_∞-*operad* if and only if each $\mathcal{C}(n)$ is contractible.

Theorem. *A connected space Y (of the homotopy type of a CW-complex with a nondegenerate base point) has the homotopy type of a based infinite loop space, i.e. has the homotopy type of $\Omega^n X_n$ for some spaces X_n for all n, if Y admits the structure of an E_∞-space.*

Boardman and Vogt were able to identify many infinite loop spaces, e.g. the classifying space BPL for PL bordisms, by use of the linear isometries operad (see section **??**), which has recently been used to great advantage in a very powerful approach to the stable homotopy category [91].

Homotopy characterization of iterated loop spaces $\Omega^n X_n$ for some space X_n for some particular n required the full power of the theory of operads (see section **??**). The relevant operad is one of the most basic topological examples, the *little n-cubes operad* originally due to Boardman and Vogt [9], $\mathcal{C}_n = \{\mathcal{C}_n(j)\}_{j\geq 1}$, where $\mathcal{C}_n(j)$ consists of an ordered collection of j n-cubes linearly embedded in the standard n-dimensional unit cube I^n with disjoint interiors and axes parallel to those of I^n.

Theorem. *A connected space Y (of the homotopy type of a CW-complex with a nondegenerate base point) has the homotopy type of a based n-fold loop space $\Omega^n X_n$ for some space X_n if Y admits the structure of a $\mathcal{C}_n$-space.*

For $n = 1$, $\mathcal{C}_1$ is an A_∞-operad of which the earlier associahedron operad $\mathcal{K}$ forms a small 'model.' Small models for the $\mathcal{C}_n$ for $n > 1$ came much later (see section **??**).

For some applications, particularly to mathematical physics, the *little n-disks operad* $\mathcal{D}_n = \{\mathcal{D}_n(j)\}_{j\geq 1}$ has some advantages. Let $B^n \subset \mathbb{R}^n$ denote the standard unit ball. The space $\mathcal{D}_n(j)$ consists of an ordered collection of j embeddings of B^n in B^n by translation and dilation with disjoint interiors.

Since an element of $\Omega^n X_n$ can be regarded as a based map $(S^n, *) \to (X_n, *)$, there is an action of the rotation group $SO(n)$ on $\Omega^n X_n$. The operad that reflects this symmetry is the *framed little n-disks operad* $\mathcal{F}_n = \{\mathcal{F}_n(j)\}_{j\geq 1}$. The space $\mathcal{F}_n(j)$ consists of an ordered collection of j embeddings of B^n in B^n by translation, rotation and dilation with disjoint interiors. This extra $SO(n)$-symmetry is relevant to the study of free loop spaces and other applications (see section **??**).

For A_∞-spaces and E_∞-spaces, we have the simple description of A_∞-non-Σ and E_∞-operads as having contractible components; for $\mathcal{C}_n$-spaces, the characterization of an operad homotopy type is more subtle (see section **??**).

1.8. A_∞-algebras.

Although introduced originally in the category of topological spaces, (non-Σ) operads were implicit almost immediately in the differential graded category. Thanks to the cellular structure of the associahedra $\mathcal{K}_n$, the cellular chain complexes $CC_*(\mathcal{K}_n)$ form an operad $\mathcal{A}_\infty$ in the category of dg $\mathbb{Z}$-modules. Algebras over $\mathcal{A}_\infty$ generalize dg associative algebras and can be defined directly.

An A_∞-*algebra* (or strongly homotopy associative algebra) consists of a graded module V with maps

$$m_n : V^{\otimes n} \to V \text{ of degree } n-2$$

satisfying suitable compatibility conditions $(A_n)_{n\geq 1}$. In particular,

(A_1) $m_1 = d$ is a differential,

(A_2) $m = m_2 : V \otimes V \to V$ is a chain map, that is, d is a derivation with respect to $m = m_2$,

(A_3) $m_3 : V^{\otimes 3} \to V$ is a chain homotopy for associativity of the multiplication m, i.e.

$$m_3 d^{\otimes 3} + dm_3 = m(m \otimes 1) - m(1 \otimes m),$$

where $d^{\otimes 3}$ denotes $d \otimes 1 \otimes 1 + 1 \otimes d \otimes 1 + 1 \otimes 1 \otimes d$,

(A_4) m_4 is a 'higher homotopy' such that $m_4 d^{\otimes 4} - dm_4$ has five terms, corresponding to the edges of the pentagon K_4:

$$m_4 d^{\otimes 4} - dm_4 = m_3(m_2 \otimes 1 \otimes 1 - 1 \otimes m_2 \otimes 1 + 1 \otimes 1 \otimes m_2) - m_2(m_3 \otimes 1 + 1 \otimes m_3).$$

An alternate formulation generalizes the bar construction for an associative differential graded algebra: An A_∞-algebra structure on a positively graded vector space A is equivalent to a coderivation differential δ of degree -1 with respect to the total grading on the tensor coalgebra $\mathbb{T}^c(\uparrow A)$ on the suspension of the graded vector space A (see section **??**). As a coderivation, δ is determined by the formula $\delta = \delta_1 + \delta_2 + \cdots$, where

$$\delta_n(\uparrow a_1 \otimes \cdots \otimes \uparrow a_n) := \epsilon \cdot \uparrow m_n(a_1 \otimes \cdots \otimes a_n), \text{ for } a_1, \ldots, a_n \in A,$$

where ϵ is an appropriate sign.

1.9. Partiality.

The definition of an operad postulates the operations γ or $\circ_i$ being defined for all elements of $\mathcal{O}(k) \times \mathcal{O}(j_1) \times \cdots \mathcal{O}(j_k)$, respectively all elements of $\mathcal{O}(k) \times \mathcal{O}(j)$, just as the n-ary operations of an algebra are usually specified to be defined for all n-tuples of elements. There are, however, may naturally occurring examples in which the relevant operations are defined only on specified subsets (compare relaxing the definition of a monoid to that of a category). Such *partial* structures are of interest at three levels.

A *partial algebra* [67] X over an operad $\mathcal{O}$ denotes a structure in which there are specified Σ_j-invariant *domains* $X_j \subset X^j$ and maps $\mathcal{O}(j) \times X_j \to X$ satisfying the appropriately restricted version of the usual axioms.

In the mathematics of vertex operator algebras (VOAs) [51, 52], the definition of a *partial operad* similarly relaxes the definition by specifying domains of *composable* elements in $\mathcal{O}(k) \times \mathcal{O}(j_1) \times \cdots \times \mathcal{O}(j_k)$ and requiring the symmetries and associativity relations hold only on such composable domains.

Yet another kind of partial operad structure on configuration spaces of points in a Riemannian manifold is considered in [83]. The Axelrod-Singer compactification [1] of the configuration space is then shown to be an appropriate operadic completion of this partial operad structure.

Just as a category can be considered as a partial algebra over the associative operad $\mathcal{A}ss$, so an A_∞-category is a partial algebra over $\mathcal{A}ss_\infty$.

An *A_∞-category* $\mathcal{D}$ is a structure having a collection of objects and spaces of morphisms for which homotopy makes sense together with structure satisfying the usual axioms of category, except that composition of morphisms is associative only up to homotopy in the strong sense. For example, for morphisms f, g, h for which $(fg)h$ and $f(gh)$ are defined, these composites are required to be homotopic in the space of morphisms, the pentagon condition must be satisfied up to homotopy, etc.

This was first done by Smirnov by 1987 [112] to handle functorial homology operations and their dependence on choices (cf. indeterminacy). More recently, Fukaya [28] reinvented A_∞-categories with remarkable applications to Morse theory and Floer homology. Inspired by this work, Nest and Tsygan have proposed an A_∞-category with automorphisms of an associative algebra as objects and for the space of morphisms, a twisted version of the Hochschild complex of the corresponding endomorphism algebras.

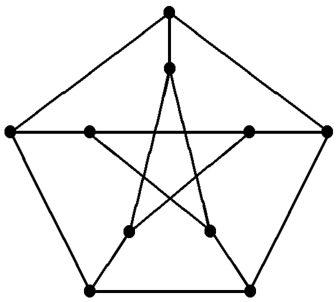
Categories can be related to classifying spaces via the realization of the nerve of the category. As A_∞-maps can be identified with maps of the corresponding classifying spaces (see section **??**), so lax functors (which are the name for A_∞-maps between categories) can be identified with maps of the corresponding realized nerves. Iterated classifying spaces appear in relation to monoidal categories, braided monoidal categories, etc. (see section **??**).

1.10. L_∞-algebras.

Perhaps because the very definition of a Lie algebra involves an additive structure and thus has no analog for spaces, the Lie analog L_∞-algebras of A_∞-algebras did not appear until the early 80's in the work of Schlessinger and Stasheff [108, 107] on perturbations of rational homotopy types, though examples abound in Sullivan's minimal models [123] in the 70's.

Just as an A_∞-algebra can be described either in terms of a family of multi-linear operations or in terms of a coderivation differential on a tensor coalgebra, so an L_∞-algebra can be described either in terms of suitably symmetric n-ary brackets or in terms of a coderivation differential on the corresponding cocommutative coassociative coalgebra $\bigwedge^c$.

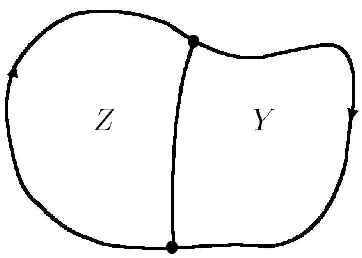
Since the Jacobi identity, consisting of three terms, does not correspond to a commutative diagram, one could not expect the axioms of L_∞-algebras be modeled on a nice sequence of polyhedra, as A_∞-algebras are modeled by the chain complex of the associahedra; the reason being roughly that there are no segments with three ends. But there is still a sequence of bipartite graphs which reflects the axioms of L_∞-algebras, introduced in [88] and called the Lie-hedra. The fourth term of this sequence is the Peterson graph:



These L_∞-algebras became explicit in the context of deformation theory where the guiding principle (going back to Grothendieck but championed by Deligne and Schlessinger-Stasheff) is that a deformation problem is 'controlled' by a dg Lie algebra or, rather, by a homotopy type of dg Lie algebra.

In the early 90's, L_∞-algebras appeared in various contexts, the most novel being that of closed string field theory as constructed by the physicist Zwiebach, where the relation to moduli spaces of punctured Riemann spheres is crucial.

As part of his closed string field theory [136], Zwiebach introduced a bracket on the differential forms on the free loop space of a manifold. The idea is that two closed strings (free loops) Y and Z join to form a third $Y * Z$ if a semi-circle of one agrees with a reverse oriented semi-circle of the other, as indicated on the following picture:



(1.4)

The loop $Y * Z$ is formed from the complementary semi-circles of each.

On functions or differential forms on the free loop space, the convolution bracket then corresponds to all ways of decomposing a free loop in the form $Y * Z$. To be more precise, the convolution product of two *fields* ϕ, ψ, i.e. complex-valued functions on the space of strings, is defined by the formula

$$(\phi * \psi)(U) := \int_{U=Z*Y} \phi(Z)\psi(Y)d\mu, \tag{1.5}$$

where the integration is taken over all decompositions of the string U into Z and Y as in (1.4), and μ is a certain measure which is in fact known to people who deal with stochastic integrals as a Brownian bridge, an analog of Brownian motion with both ends of the path fixed.

It can be shown that the product (1.5) is a bracket of a certain L_∞ structure whose higher homotopies can be described by similar explicit formulas over some more complicated configuration spaces.

The operad $\mathcal{L}ie_\infty$ for L_∞ algebras appeared independently about the same time [40, 50]. Surprisingly, the operad for ordinary Lie algebras appears to have occurred first in [50], though it is clearly implicit in [17]. Hinich and Schechtman give a description of the operad for L_∞-algebras in the language of trees, but attribute the discovery to Ginzburg and Kapranov, who moreover give the operad a very nice geometric interpretation which clarifies the relation to physics via (a stratification of) a real compactification of moduli spaces of Riemann spheres with punctures (see section **??**). Ginzburg and Kapranov also put it in the context of the Koszul duality for operads they developed [40] (see section **??**).

1.11. C_∞-algebras.

Just as we have both A_∞-spaces and A_∞-algebras, so too we have E_∞-spaces and E_∞-algebras. These are algebras over an operad $\mathcal{O}$ for which the components $\mathcal{O}(j)$ are Σ_j-free and contractible dg modules, which are homotopy commutative and have all sorts of higher homotopies.

For algebras over an A_∞-operad, the small representative provided by the associahedra or their cellular chains has been very useful. We have no such small representatives of E_∞-operads and so no nice small set of higher homotopies for commutativity, though some explicit constructions of algebraic E_∞-operads were given [113].

An interesting alternative is to consider strictly commutative A_∞-algebras. In that case, what symmetry should we require of the homotopies for associativity? There are 'standard constructions' which provide adjoint functors from dg Lie algebras to cocommutative dg coalgebras and from commutative dg algebras to dg Lie coalgebras ([101, 93], see also the tables in Section 1.13). For dg Lie algebras L, the standard construction is the graded symmetric coalgebra on the suspension of L and thus the n-ary brackets of an L_∞-algebra have corresponding symmetry. The standard construction for dg commutative algebras is the Harrison chain complex, which is a dg Lie coalgebra (see section **??**). This can be described, using a result of Ree [103], as the quotient of the tensor coalgebra by the 'shuffle decomposables.' Thus an appropriate specification for a 'strictly commutative' A_∞-algebra A, to be called a *C_∞-algebra* (also known as a balanced A_∞-algebra), is via a coderivation on the graded Lie coalgebra on the suspension of A or equivalently in terms of a coherent set of n-ary products which vanish on shuffles.

1.12. n-ary algebras.

Although we have emphasized that operads parameterize families of operations of various 'arities,' all the examples considered so far start with binary operations. On the other hand, there are generalizations of binary algebras which are generated by a single n-ary operation. Remarkably, in the case of Lie algebras, such generalizations have received considerable attention, beginning with Filipov in 1985 and continuing, independently, by Hanlon and Wachs [43] (combinatorial algebraists), by Azcárraga and Bueno [2] (physicists), and by Gnedbaye [41] who is the only one to treat them operadically. *n-Lie algebras* have an alternating (or graded skew symmetric in the graded case) n-multilinear bracket $[-,\dots,-]$ which satisfies

$$\sum_{\sigma\in\Sigma_{2n-1}} (-1)^{\mathrm{sgn}(\sigma)}[v_{\sigma(1)}, v_{\sigma(2)}, \dots, v_{\sigma(n-1)}, [v_{\sigma(n)}, \dots, v_{\sigma(n-1)}]] = 0.$$

For $n = 2$, the above equation is the standard Jacobi identity, thus 2-Lie algebras are ordinary Lie algebras. The above axiom is, up to degree and signs, that of an L_∞-algebra with only a single n-ary operation nonzero.

On the other hand, Nambu invented, as a 'toy model' for quarks considered as triples, a trilinear bracket given, for smooth functions f_1, f_2, f_3 on $\mathbb{R}^3$, by the Jacobian:

$$\{f_1, f_2, f_3\} := \frac{\partial(f_1, f_2, f_3)}{\partial(x_1, x_2, x_3)} = \sum_{\sigma\in\Sigma_3} (-1)^{\mathrm{sgn}(\sigma)} \frac{\partial f_1}{\partial x_{\sigma(1)}} \frac{\partial f_2}{\partial x_{\sigma(2)}} \frac{\partial f_3}{\partial x_{\sigma(3)}}$$

This skew-symmetric trilinear product satisfies another generalization of Jacobi identity, namely

$$\{\{f_1, f_2, f_3\}, f_4, f_5\} + \{f_3, \{f_1, f_2, f_4\}, f_5\} + \{f_3, f_4, \{f_1, f_2, f_5\}\} = \{f_1, f_2, \{f_3, f_4, f_5\}\}.$$

This was further generalized by Takhtajan [124] to the *fundamental identity* for the generalized Nambu n-linear 'bracket.' This identity was known also to Flato and Fronsdal in 1992, though unpublished, and to Sahoo and Valsakumar [105].

For a comparison of these two distinct generalizations of the Jacobi identity for n-ary brackets, see [128]. Other examples of n-ary algebras are introduced in [41].

1.13. Operadic bar construction and Koszul duality.

Since, as we will see in **??**, a dg-operad $\mathcal{O}$ is itself a monoid in a symmetric monoidal category, the bar construction expressed in the appropriate generality applies to $\mathcal{O}$, producing a dg co-operad $\mathcal{BO}$ (see section ??). The linear dual of $\mathcal{BO}$ is then an operad which we denote $\Omega\mathcal{O}$. The co-operad $\mathcal{BO}$ was introduced by Ginzburg and Kapranov in [40] and we call it the *bar construction* on the operad $\mathcal{O}$. The operad $\Omega\mathcal{O}$ is then the *dual bar construction* on the operad $\mathcal{O}$.

If $\mathcal{A}ss$ is the operad for associativity, then $\Omega\mathcal{A}ss$ is an operad $\mathcal{A}ss_\infty$ for A_∞-algebras and the homology $H(\Omega\mathcal{A}ss)$ is again the operad $\mathcal{A}ss$ for associative algebras. If $\mathcal{L}ie$ is the operad for Lie algebras, then $\Omega\mathcal{L}ie$ is the operad $\mathcal{C}om_\infty$ for C_∞ algebras and $H(\Omega\mathcal{L}ie) = \mathcal{C}om$, the operad for commutative algebras. Similarly, the dual bar construction on the operad $\mathcal{C}om$ is the operad $\mathcal{L}ie_\infty$ for L_∞-algebras and the cohomology $H(\Omega\mathcal{C}om)$ is the operad $\mathcal{L}ie$. We may systemize the above facts as:

$$\begin{aligned}
&\text{if } \mathcal{O} = \mathcal{A}ss, \text{ then } \Omega\mathcal{O} = \mathcal{A}ss_\infty \text{ and } H(\Omega\mathcal{O}) = \mathcal{A}ss\\
&\text{if } \mathcal{O} = \mathcal{L}ie, \text{ then } \Omega\mathcal{O} = \mathcal{C}om_\infty \text{ and } H(\Omega\mathcal{O}) = \mathcal{C}om, \text{ and}\\
&\text{if } \mathcal{O} = \mathcal{C}om, \text{ then } \Omega\mathcal{O} = \mathcal{L}ie_\infty \text{ and } H(\Omega\mathcal{O}) = \mathcal{L}ie.
\end{aligned} \tag{1.6}$$

A similar pattern was observed in rational homotopy theory, where the following correspondence has been folklore:

associative algebra products $-\cdot-$ on a vector space A	$\longleftrightarrow$	square zero coderivations δ on the tensor coassociative coalgebra $\mathbb{T}^c(\uparrow A)$ with $\delta(\uparrow a \otimes \uparrow b) = (-1)^{\deg(a)+1} \uparrow(a\cdot b)$
commutative associative algebra products $-\cdot-$ on a graded vector space A	$\longleftrightarrow$	square zero coderivations δ on the Lie coalgebra $\mathbb{L}^c(\uparrow A)$ with $\delta([\uparrow a, \uparrow b]) = (-1)^{\deg(a)+1} \uparrow(a\cdot b)$
Lie algebra brackets $[-,-]$ on a graded vector space A	$\longleftrightarrow$	square zero coderivations δ on the cocommutative coassociative coalgebra $\wedge^c(\uparrow V)$ with $\delta(\uparrow a \wedge \uparrow b) = (-1)^{\deg(a)+1} \uparrow[a,b]$

On a more sophisticated level, the above pattern manifests itself in the $\mathcal{L}_*$ - $\mathcal{C}^*$ duality between Quillen and Sullivan models of a topological space or, even more abstractly, by the equivalence between the homotopy category of differential graded Lie algebras and the homotopy category of differential graded commutative algebras [126].

A conceptual, operadic explanation for this pattern was provided by Ginzburg and Kapranov by their concept of quadratic (sometimes also called Koszul) duality for operads which defines, for any quadratic operad $\mathcal{O}$, its *quadratic dual* $\mathcal{O}^!$.

Quadratic operads are defined as those having a presentation of a special type (see **??**). As we will see later, all three operads $\mathcal{A}ss$, $\mathcal{L}ie$ and $\mathcal{C}om$ are quadratic, thus their duals are defined and they are

$$\mathcal{A}ss^! = \mathcal{A}ss,\ \mathcal{C}om^! = \mathcal{L}ie \text{ and } \mathcal{L}ie^! = \mathcal{C}om.$$

Ginzburg and Kapranov went further and introduced a certain homological property of quadratic operads. Operads sharing this property are called *Koszul* (quadratic) operads and are characterized by the isomorphism $H(\mathcal{O}) = \mathcal{O}^!$. So Table (1.6) above expresses the fact that all three operads $\mathcal{A}ss$, $\mathcal{L}ie$ and $\mathcal{C}om$ are Koszul.

The concept of quadratic dual and of Koszulness was inspired by a similar classical concept of Priddy for associative algebras, which is in accord with the opening remark of this section that operads are associative monoids in a suitable category.

Of course, mention of the bar construction should bring to mind the possibility of defining the 'homology of an operad' by analogy with Hochschild homology of an associative algebra. This was first considered by Markl in the context of PROPs [80] as a tool for the study of deformations of general algebraic structures, but the theory was not fully developed until the advent of Koszulness for operads as special cases of PROPs, which made explicit calculations possible.

As suggested by [88], the homology of an operad is closely related to the coherence of corresponding algebraic or categorial structures. A closed model category structure on the category of operads was studied by Hinich [48]. The work of Baues, Jibladze and Tonks [5] then introduced a homology theory for monoids in a monoidal category of which operads are special cases.

1.14. Cyclic operads.

Algebras with invariant inner products $\langle -, - \rangle$ are of considerable importance in mathematics and especially in mathematical physics; invariance means that $\langle ab, c\rangle = \langle a, bc\rangle$ in the associative case and $\langle [a,b], c\rangle = \langle a, [b,c]\rangle$ in the Lie case. Using the inner product, the n-ary operations $A^{\otimes n} \to A$ can be converted to operations $A^{\otimes n+1} \to \mathbf{k}$ with cyclic symmetry. To handle such algebras via operads, the notion of a cyclic operad was introduced by Getzler and Kapranov [37] following ideas of Kontsevich [63]. The formal definitions and applications are given in section **??**, but the basic idea is well illustrated by the planar tree operad and by the associahedra. If we treat the root of a planar tree as just another leaf, then the order of leaves is now cyclic rather than linear. We no longer have a grafting operation $\circ_i$ but rather can define ${}_i\circ_j$ by grafting the i-th leaf of the first tree to the j-th leaf of the second tree and renumbering accordingly. It is obvious that the pentagon admits an action by the cyclic group $\mathbb{Z}/5\mathbb{Z}$ and an alternate view of K_5 shows a 6-fold symmetry. Recall ?? that the associahedra form a non-Σ-operad $\mathcal{K}$ which generates an operad $\mathcal{KK}$ freely: $\mathcal{KK}(j) = \Sigma_j \times K_j$. Corresponding to the cyclic symmetry of unrooted planar trees with $j+1$ leaves and the visible cyclic symmetry of K_4 and K_5, the associahedra can be regarded as forming a cyclic non-Σ-operad.

Now consider an A_∞-algebra with an inner product and convert the n-ary operations $A^{\otimes n} \to A$ to operations $A^{\otimes n+1} \to \mathbf{k}$ with cyclic symmetry. These cyclic A_∞-algebras, as well as their Lie and commutative analogs, play a key role in Kontsevich's formal noncommutative symplectic geometry via his graph cohomology (see section ??). They also make an appearance in the algebra of string field theory.

1.15. Moduli spaces and modular operads.

One reason for the explosive development of operad theory in the 90's was the introduction of operadic structures in topological field theories, e.g. CFT's (conformal field theories) and SFT's (string field theories), which in turn was inspired by the importance of moduli spaces of Riemann surfaces with punctures or boundaries (or other decorations) in these physical theories. A major tool in the study of such moduli spaces is the combinatorial structure of decompositions of Riemann surfaces into elementary pieces, such as a Riemann sphere with holes. The combinatorics can be described simply in terms of a graph so that a tubular neighborhood of the graph is topologically the surface. If the graph is a tree, the corresponding surface has genus 0; physicists will speak of a theory at tree level when it is governed by a moduli space of Riemann spheres with boundaries.

Since trees play such a fertile role in the theory of operads, it is perhaps not surprising that there is a generalization called a 'modular operad' for which graphs provide the basic combinatorial structure [38].

The components of an operad are indexed by non-negative integers, those of a modular operad by pairs (g, n) where g is a map from the vertices of the graph to the natural numbers. In other words, g labels each vertex of the graph by a natural number, called the *genus* of the vertex. The labels are subject to the condition that at each vertex v, we have $2(g(v)-1)+val(v) > 0$ where val denotes the valence of the vertex, the number of (half) edges incident with the vertex. The analog of 'leaves' for trees are the legs of a graph, i.e. half-edges with a vertex at only one end.

If the legs of a labelled graph are labelled by a finite set S, the grafting operation for trees can be extended to graphs, with the additional subtlety that two legs of a single graph are allowed to be grafted to each other. The formal definition of a modular operad based on these operations is given in section ??. They play a central role in the physical theories mentioned above and considered in detail in section ??. That same 'down to earth' combinatorial quality reflected in the study of moduli spaces appeared earlier in Grothendieck's 'Esquisse' (see section ??).

1.17. From topological operads to dg operads.

For a topological operad $\mathcal{O}$, chain complexes which respect products (functors from topological spaces to dg modules taking products to products) provide a *chain operad*, an operad in the category of dg modules over the ground ring, with the monoidal structure given by the tensor product. For example, the singular chain complex always works. Given a chain operad $\mathcal{O}$, the corresponding homologies $H(\mathcal{O}(j))$ form an operad $H(\mathcal{O}(j))$ in the category of graded modules.

Since the associahedra are themselves cells or, rather, regular cell complexes with the operad structure given by the cellular inclusions $\circ_i : K_r \times K_s \to K_{r+s-1}$, their cellular chain complexes $\{CC_*(\mathcal{K}_n\}$ form a chain operad, which is precisely the operad $\mathcal{A}ss_\infty$ for A_∞-algebras, while the homology $H(\mathcal{K})$ is the operad for associative algebras.

For any E_∞-operad, the homology operad is that for commutative associative algebras. The calculation for the little cubes operad is not so simple, but is known, thanks to the calculations of Fred Cohen [16]. As a space, $\mathcal{C}_n(j)$ has the homotopy type of the configuration space $F(\mathbb{R}^n, j)$ of ordered j-tuples of points in $\mathbb{R}^n$.

Theorem. *An algebra over the operad $H(\mathcal{C}_n)$ is a graded algebra with two operations:*

(i) a graded commutative associative product, $a \otimes b \mapsto ab$, and
(ii) a graded anticommutative 'bracket' of degree $n-1$, $a \otimes b \mapsto [a, b]$

such that, after regrading, the bracket satisfies the usual graded Jacobi identity and the two operations are related by a graded Leibniz rule:

$$[a, bc] = [a, b]c + (-1)^{(\deg(a)+n+1)\deg(b)} b[a, c].$$

Some authors refer to this bracket structure as an 'n-Lie algebra'; unfortunately the same phrase is used for generalizations of Lie algebra in terms of a single n-ary 'bracket' [128], as mentioned above.

An algebra over the full homology $H(\mathcal{C}_n)$ is known as an *n-braid algebra* because of the relation of the homology to the braid group. It is a graded analog of a Poisson algebra.

The original work on operads and comp algebras came together unexpectedly in 1994 when it was pointed out by Getzler [?] that the Gerstenhaber algebra structure on the Hochschild cohomology of a commutative associative algebra A with coefficients in itself was an algebra over $H(\mathcal{C}_2)$. Nowadays any algebra over $H(\mathcal{C}_2)$, i.e. an algebra with the above structure for $n = 2$ is called a *Gerstenhaber algebra*. The situation at the chain level is much more subtle and will be discussed in sections **??** and **??**.

There is another way to construct dg operads from topological ones. Axelrod and Singer constructed in [1] a real compactification $\overline{F}_n(j)$ of the moduli space of configurations of j distinct points in $\mathbb{R}^n$. The collection $\{\overline{F}_n(j)\}_{j\geq 1}$ is known to be an operad in the category of manifolds with corners [83]. It has a natural stratification with strata labeled by trees, and the E_1-term of the spectral sequence of this stratification is isomorphic to $\Omega(\mathcal{B}_n^!)$, the dual bar construction on the quadratic dual $\mathcal{B}_n^!$ of the operad $\mathcal{B}_n$ for n-braid algebras. The dual bar operad $\Omega(\mathcal{B}_n^!)$ contains the operad $\mathcal{L}ie_\infty$ as a suboperad generated by the quadratic dual of $\mathcal{C}om \subset \mathcal{B}_n$.

Thus, although there is no known topological operad with chains that give the $\mathcal{L}ie_\infty$ operad, the operad $\mathcal{L}ie_\infty$ is still a suboperad of a dg operad with a geometrical origin.

1.18. Homotopy invariance in algebra and topology.

Version of March 28, 2000

As mentioned in **??**, operads were prefigured in the early investigations of loop spaces, topological groups and monoids in homotopy invariant terms. Boardman and Vogt's groundbreaking book "Homotopy Invariant Algebraic Structures on Topological Spaces" [**?**] begins with a brief history starting (in their Introduction) with Mac Lane's seminar at the University of Chicago in 1967. A detailed account of this seminar and its seminal influence has been given by Vogt [**?**], who kept excellent notes. They remark:

> *'The disadvantage of topological groups and monoids is that they do not live in homotopy theory,'*

meaning that a space homotopy equivalent to a topological monoid need not admit a strictly associative multiplication (with unit) recognized by the equivalence. Stasheff's A_∞-spaces did live in homotopy theory; indeed, this was a major reason for studying them.

Boardman and Vogt developed their theory initially to establish various stable groups as infinite loop spaces, but soon expanded their research to more general homotopy invariant algebraic structures. Their book is written in the language of 'categories of operators' called PROPs. Operads give rise to a special class of PROPs and serve much the same purpose for the main applications.

Although the initial emphasis was on homotopy theory for topological spaces, from the very beginning there was implicit the study of homotopy invariant algebraic structures in other contexts. The appropriate setting for describing algebraic structures is now acknowledged to be that of a monoidal category **??**, while for homotopy theory it is that of a (closed) model category (CMC). We are content to work with (compactly generated) topological spaces or dg modules (chain complexes).

Since there is a simple operad $\mathcal{M}on$ describing topological monoids, with $\mathcal{M}on(n) = \Sigma_n$, $n \geq 1$, (except for $\mathcal{M}on(0)$ which is a singleton) and the more complicated $\mathcal{K}$ for the homotopy invariant version of monoids, called A_∞-spaces, an operad must have some special properties to capture an algebraic structure homotopy invariantly. Somewhat analogously, for an associative algebra (without unit) the corresponding operad is $\mathcal{A}ss$ with $\mathcal{A}ss(n) = k[\Sigma_n]$, $n \geq 1$, with the more complicated $\mathcal{A}_\infty$ for A_∞-algebras, see **??**.

Given an operad $\mathcal{P}$ for an (ordinary) algebraic theory, a *strongly homotopy $\mathcal{P}$-algebra* is to be a homotopy invariant concept. By this we mean that each 'space' (which may be either a topological space or a chain complex, depending on the category we work in), homotopy equivalent to a $\mathcal{P}$-space, admits a strongly homotopy $\mathcal{P}$-structure recognized by the equivalence. In particular, each $\mathcal{P}$-space is of course also a strongly homotopy $\mathcal{P}$-space. From our modern perspective, a conceptual understanding of homotopy invariance is the following.

There exists, at least 'philosophically,' a closed model category (CMC) structure on the category of operads, both on the topological and on the algebraic side, though rigorous constructions of these closed model structures have been given only in some special cases. A strongly homotopy $\mathcal{P}$-algebra is now understood to be an algebra over a *cofibrant model* of $\mathcal{P}$. With respect to such a CMC structure, a cofibrant model of $\mathcal{P}$ is, by definition, a cofibrant operad $\mathcal{C}$ together with a weak equivalence (again in the CMC structure above) $\alpha : \mathcal{C} \to \mathcal{P}$. Homotopy invariance is then expressed by the principle:

'Algebras over cofibrant operads are homotopy invariant.'

On the topological side, this principle is implicit in the above mentioned book by Boardman and Vogt [10], where a cofibrant resolution was provided by the W-construction $W\mathcal{P}$. The cofibrant nature of $W\mathcal{P}$ was clarified much later, in [131]. On the algebraic side, the analogous results were worked out in [85]. While a general cofibrant model is unique only up to a weak equivalence, in the category of chain complexes there exist special cofibrant operads, called minimal operads, having the wonderful property that they are *isomorphic* if and only if and only if they are weakly equivalent **??**. A cofibrant model of $\mathcal{P}$ using this special kind of cofibrant operads is called the *minimal model* of $\mathcal{P}$ [?].

In the category of chain complexes (= differential graded modules), all classical strongly homotopy algebras are algebras over these minimal cofibrant operads, so they are homotopy invariant in this category. See also Section **??**.

Part II

Chapter 1

Operads in a symmetric monoidal category

1.1. Symmetric Monoidal Categories.

In this section we develop some of the tools necessary for the study of operads in full generality, beginning with some basic definitions and results from category theory [14, 15].

A *category with multiplication* is a category $\mathcal{C}$ and a covariant bifunctor $\odot : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$. The multiplication is associative if there is an *associator* or *associativity constraint*, that is, an invertible natural transformation

$$a : \odot(id \times \odot) \to \odot(\odot \times id), \; a_{A,B,C} : A \odot (B \odot C) \to (A \odot B) \odot C \text{ for } A, B, C \in \mathcal{C}, \tag{1.1}$$

where as usual, id denotes the identity functor, $A \odot (B \odot C)$ denotes $\odot(id \times \odot)(A, B, C)$, etc.

Having such an associator, we can consider diagrams whose vertices are iterates of $\odot$ and whose edges are applications of instances of a. Then $(\mathcal{C}, \odot)$ is called *coherent* if all these diagrams commute, the simplest being the pentagon (see Figure 1.1). Mac Lane [15] proved that commutativity of the pentagon is a necessary and sufficient condition for coherence.

A category $\mathcal{C}$ is called *monoidal* if it has a multiplication $\odot$ with an associator a satisfying the Mac Lane pentagon condition and there is a preferred object $\mathbb{1} \in \mathcal{C}$ called the *unit* object together with natural isomorphisms

$$l = l_A : \mathbb{1} \odot A \to A, \quad r = r_A : A \odot \mathbb{1} \to A$$

such that, in addition to the pentagon condition on a, the triangles in Figure 1.2 commute. It is an easy exercise (see [57]) that commutativity of the triangle on the right together with the naturality of l and r imply the commutativity of the other two triangles. A monoidal category is called *strict* when all the $a_{A,B,C}$, l_A and r_A are identity morphisms. One formulation of Mac

$$\begin{tikzcd}[column sep=small]
 & & (A \odot B) \odot (C \odot D) \arrow[drr] & & \\
A \odot (B \odot (C \odot D)) \arrow[urr] \arrow[dr] & & & & ((A \odot B) \odot C) \odot D \\
 & A \odot ((B \odot C) \odot D) \arrow[rr] & & (A \odot (B \odot C)) \odot D \arrow[ur] &
\end{tikzcd}$$

Figure 1.1: The Pentagon

$$\begin{tikzcd}
A \odot (B \odot \mathbb{1}) \arrow[r, "a"] \arrow[dr, "id \odot r"'] & (A \odot B) \odot \mathbb{1} \arrow[d, "r"] \\
 & A \odot B
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{1} \odot (A \odot B) \arrow[r, "a"] \arrow[dr, "l"'] & (\mathbb{1} \odot A) \odot B \arrow[d, "l \odot id"] \\
 & A \odot B
\end{tikzcd}$$

$$\begin{tikzcd}
A \odot (\mathbb{1} \odot B) \arrow[r, "a"] \arrow[dr, "id \odot l"'] & (A \odot \mathbb{1}) \odot B \arrow[d, "r \odot id"] \\
 & A \odot B
\end{tikzcd}$$

Figure 1.2: The coherence conditions on the unit. Actually, the condition at the bottom suffices.

Lane's coherence theorem is that any monoidal category is equivalent to a strict monoidal category. Stated in this form, Mac Lane's theorem is sometimes called a rectification theorem.

A monoidal category is called *symmetric* if there exists a natural transformation

$$s_{A,B} : A \odot B \longrightarrow B \odot A,$$

which is of order two, $s_{B,A}s_{A,B} = id_{A \odot B}$, and satisfies the hexagon identity, which is equivalent to the commutativity of the diagram in Figure 1.3.

Remark 1.1.1. There is an inconsistency in the notation in the literature, as to whether the associator should go from $A \odot (B \odot C) \to (A \odot B) \odot C$ or vice versa, but by the invertibility assumption, it does not matter which we chose. However, the different choices imply alternative forms of the pentagon identity. Mac Lane uses the symbol a for associativity,

$$
\begin{tikzcd}
& A \odot (B \odot C) \arrow[dl, "id_A \odot s_{B,C}"'] \arrow[dr, "a"] & \\
A \odot (C \odot B) \arrow[d, "a"'] & & (A \odot B) \odot C \arrow[d, "s_{A \odot B, C}"] \\
(A \odot C) \odot B \arrow[dr, "s_{A,C} \odot id_A"'] & & C \odot (A \odot B) \arrow[dl, "a"] \\
& (C \odot A) \odot B &
\end{tikzcd}
$$

Figure 1.3: The Hexagon

and uses this to mean that the parentheses are moved to the left. Drinfel'd uses the symbol Φ to mean the operation of moving the parentheses to the right.

The structure of *braided monoidal category* generalizes that of symmetric monoidal category by allowing a symmetry which is not necessarily order two. There is another hexagon identity which is given by commutativity of a diagram similar to Figure 1.3 with $s_{A \odot B,C}$, $s_{B,C}$, and $s_{A,C}$ replaced by $s^{-1}_{C,A \odot B}$, $s^{-1}_{C,B}$, and $s^{-1}_{C,A}$, respectively.

In the following discussion we will focus on two main examples: first, the category $\mathtt{Mod}_{\mathbf{k}}$ of $\mathbf{k}$-modules for a commutative ring $\mathbf{k}$ and more generally, differential graded $\mathbf{k}$-modules, second, the category $\mathtt{Set}$ of sets and its specialization to topological spaces. The category $\mathtt{Mod}_{\mathbf{k}}$ has a monoidal structure arising from the standard tensor product over $\mathbf{k}$,

$$A \odot B := A \otimes_{\mathbf{k}} B \text{ for } A, B \text{ objects of } \mathtt{Mod}_{\mathbf{k}} \text{ and } \mathbb{1} := \mathbf{k}.$$

The monoidal structure on $\mathtt{Set}$ is given by the cartesian product

$$X \odot Y := X \times Y \text{ for } X, Y \text{ objects of } \mathtt{Set} \text{ and } \mathbb{1} := \{x\}, \text{ the set with one element.}$$

The subcategory of the category of sets consisting of compactly generated Hausdorff spaces and continuous maps has a monoidal structure given by the cartesian product with the compactly generated topology. This was the setting in which operads were first defined by May, [92].

In all the cases in the previous paragraph, the associator and the isomorphisms in the definition of the left and right units are given by the standard identifications. The symmetry is given by transposing factors. We point out that although these examples are not strictly monoidal, the identifications are canonical, so these categories are coherent. An example of a strict monoidal category is given by the category of endofunctors on a given category with natural transformations as morphisms and composition as the monoidal structure.

To define a monoidal structure on differential graded $\mathbf{k}$-modules, $\mathtt{dgMod}_{\mathbf{k}}$, extend the tensor product of $\mathbf{k}$-modules to a tensor product of complexes in the standard way:

$$\begin{aligned}(A_* \odot B_*)_k &:= \bigoplus_{i+j=k} A_i \otimes_{\mathbf{k}} B_j, \\ d_{A\odot B}(a \otimes b) &:= d_A(a) \otimes b + (-1)^{deg(a)} a \otimes d_B(b)\end{aligned}$$

for $(A_*, d_A), (B_*, d_B)$ objects of $\mathtt{dgMod}_{\mathbf{k}}$ and unit object,

$$\mathbb{1}_j := \begin{cases} \mathbf{k} \text{ when } j = 0 \\ 0 \text{ when } j \neq 0. \end{cases}$$

A monoidal category is precisely the setting for the categorical concept of a monoid or "algebra" in Mac Lane's original terminology, [14]. Given a monoidal category $\mathcal{C}$ as above, a $\mathcal{C}$*-monoid* consists of an object A and $\mathcal{C}$-morphisms $\gamma : A \odot A \to A$, $\eta : \mathbb{1} \to A$ (respectively called the "multiplication" and "unit" for A) which satisfy associativity and unitarity conditions:

$$\gamma(id_A \odot \gamma) = \gamma(\gamma \odot id_A)a_{A,A,A}, \quad \gamma(id_A \odot \eta) = r_A, \quad \gamma(\eta \odot id_A) = l_A. \tag{1.2}$$

The definition of morphisms of $\mathcal{C}$-monoids is the usual one; they must be compatible with all of the morphisms in the definition of a $\mathcal{C}$-monoid. We use $\mathtt{Mon}(\mathcal{C})$ to denote the resulting category.

Let $\mathcal{S}$ be the category of non-empty finite sets and bijections. The category $\mathcal{S}$ is equivalent to its skeleton, a small category $\widehat{\mathcal{S}}$, called the symmetric groupoid. The objects of $\widehat{\mathcal{S}}$ are the sets $[n] = \{1, 2, \cdots, n\}$, indexed by the natural numbers and the morphisms are naturally identified with the symmetric groups

$$End_{\widehat{\mathcal{S}}}([n]) = \Sigma_n.$$

1.2. Operads.

May's original definition of an operad [92] was for the category of topological spaces, with compactly generated topology, but it generalizes to any symmetric monoidal category.

Let $\mathcal{C}$ be such a symmetric monoidal category with multiplication denoted by $\odot$. Assume that $\mathcal{C}$ has small limits and colimits and that for any object A, the endofunctor $A \odot -$ preserves colimits. Let $\mathtt{Fun}(\mathcal{S}^{op}, \mathcal{C})$ be the category of contravariant functors from $\mathcal{S}$ to $\mathcal{C}$ and $\mathtt{Fun}(\widehat{\mathcal{S}}^{op}, \mathcal{C})$ be the category of contravariant functors from $\widehat{\mathcal{S}}$ to $\mathcal{C}$. Note that for each A in $\mathtt{Fun}(\mathcal{S}^{op}, \mathcal{C})$ and $X \in \mathcal{S}$, the object $A(X)$ in $\mathcal{C}$ has a right $End_{\mathcal{S}}(X)$-action. The objects in $A \in \mathtt{Fun}(\widehat{\mathcal{S}}^{op}, \mathcal{C})$ are represented by sequences of objects, $\{A(n) | n \geq 1\}$ in $\mathcal{C}$ with a right Σ_n action on $A(n)$.

Definition 1.2.1. *(Operad - May's original version) An* **operad** *in a symmetric monoidal category* $\mathcal{C}$ *is an object* $A \in \mathtt{Fun}(\widehat{\mathcal{S}}^{op}, \mathcal{C})$ *together with a family of morphisms*

$$\gamma_{n,m_1,\ldots,m_n} : A(n) \odot A(m_1) \odot \ldots \odot A(m_n) \longrightarrow A(m_1 + \cdots + m_n),$$

satisfying the axioms:

1. *Associativity. Let* m_i *be natural numbers defined for* $1 \le i \le n$*, and* $l_{i,j}$ *other numbers defined for* $1 \le i \le n$ *and* $1 \le j \le m_i$*. Define natural numbers,* $m := \sum_{1\le i\le n} m_i$*,* $l_i := \sum_{1\le j\le m_i} l_{i,j}$ *and* $l := \sum_{1\le i\le n} l_i$ *and tuples* $\mathbf{m} := (m_1, \cdots, m_n)$*,* $\mathbf{l} := (l_{1,1}, \ldots, l_{n,m_n})$*,* $\mathbf{l}_i := (l_{i,1}, \cdots, l_{i,m_i})$ *and* $\mathbf{l}' := (l_1, \cdots, l_n)$*. Let*

$$A[\mathbf{m}] := A(m_1) \odot \cdots \odot A(m_n),\ A[\mathbf{l}_i] := A(l_{i,1}) \odot \cdots \odot A(l_{i,m_i}),$$
$$A[\mathbf{l}] := A(l_{1,1}) \odot \cdots \odot A(l_{n,m_n}) = A(\mathbf{l}_1) \odot \cdots \odot A(\mathbf{l}_n) \text{ and}$$
$$A[\mathbf{l}'] := A(l_1) \odot \cdots \odot A(l_n).$$

The following diagram commutes

$$\begin{tikzcd}
A(n) \odot A[\mathbf{m}] \odot A[\mathbf{l}] \arrow[r, "\rho"] \arrow[ddd, "\gamma_{n,l_1,\ldots,l_n}"] & A(n) \odot A(m_1) \odot A[\mathbf{l}_1] \odot \cdots \odot A(m_n) \odot A[\mathbf{l}_n] \arrow[d, "id \odot \bigodot_{i=1}^{n} \gamma_{m_i,l_{i,1},\ldots,l_{i,m_i}}"] \\
& A(n) \odot A[\mathbf{l}'] \arrow[dd, "\gamma_{n,m_1,\ldots,m_n}"] \\
& \\
A(m) \odot A[\mathbf{l}] \arrow[r, "\gamma_{m,l_{1,1},\ldots,l_{n,m_n}}"] & A(l)
\end{tikzcd} \tag{1.3}$$

where ρ *applies the symmetry in* $\mathcal{C}$ *to permute the factors in* $A[\mathbf{m}] \odot A[\mathbf{l}]$ *to give a product of* $A(m_i) \odot A[\mathbf{l}_i]$*.*

2. *Equivariance. Given* $\sigma \in \Sigma_n$ *and an* n*-tuple* $\mathbf{m} = (m_1, \ldots, m_n)$*, define* $\sigma\mathbf{m} := (m_{\sigma^{-1}(1)}, \ldots, m_{\sigma^{-1}(n)})$*, and let* $\overline{\sigma} : A[\mathbf{m}] \to A[\sigma\mathbf{m}]$ *be the permutation of the factors given by the symmetry in* $\mathcal{C}$*. Then the following diagram commutes:*

$$\begin{tikzcd}
A(n) \odot A[\mathbf{m}] \arrow[r, "id \odot \overline{\sigma}"] \arrow[d, "\sigma \odot id"'] & A(n) \odot A[\sigma\mathbf{m}] \arrow[d, "\gamma_{n,m_{\sigma^{-1}(1)},\ldots,m_{\sigma^{-1}(n)}}"] \\
A(n) \odot A[\mathbf{m}] \arrow[r, "\gamma_{n,m_1,\ldots,m_n}"] & A(m)
\end{tikzcd} \tag{1.4}$$

3. *Unit. If* $\mathbb{1}$ *is the unit object of* $\mathcal{C}$*, then there is an* $\eta : \mathbb{1} \to A(1)$ *such that the composite morphisms*

$$id \odot \eta^{\odot n} \ :\ A(n) \odot \mathbb{1}^{\odot n} \to A(n) \odot A(1) \odot \cdots \odot A(1) \xrightarrow{\gamma_{n,1,\ldots,1}} A(n)$$
$$\eta \odot id \ :\ \mathbb{1} \odot A(m) \longrightarrow A(1) \odot A(m) \longrightarrow A(m)$$

are respectively the iterated right unit morphism and the left unit morphism for the underlying monoidal category $\mathcal{C}$*.*

Operads form a subcategory of $\mathtt{Fun}(\widehat{\mathcal{S}}^{op}, \mathcal{C})$ with morphisms required to commute with the morphisms defining the operad structure.

It will be useful to have operads defined in terms of the category $\mathtt{Fun}(\mathcal{S}^{op}, \mathcal{C})$, that is, dealing with finite sets instead of natural numbers. This extension presents no problem since there are functors R (restriction) and E (extension) whose composition in either order is the identity functor. Thus the categories $\mathtt{Fun}(\mathcal{S}^{op}, \mathcal{C})$ and $\mathtt{Fun}(\widehat{\mathcal{S}}^{op}, \mathcal{C})$ are isomorphic.

The restriction functor $R : \mathtt{Fun}(\mathcal{S}^{op}, \mathcal{C}) \to \mathtt{Fun}(\widehat{\mathcal{S}}^{op}, \mathcal{C})$ is defined in the obvious way. The extension functor $E : \mathtt{Fun}(\widehat{\mathcal{S}}^{op}, \mathcal{C}) \to \mathtt{Fun}(\mathcal{S}^{op}, \mathcal{C})$ is more subtle. An explicit description, which we will need later, is given in the appendix to this section. To simplify notation we shall denote $EA(X)$ by $A(X)$.

Considering our two basic examples, in the case of $\mathrm{Mod}_{\mathbf{k}}$, $A(X)$ is the module of Σ_n covariants:

$$A(X) = \left(\bigoplus_{f : X \tilde{\to} [n]} A(n) \right)_{\Sigma_n} .$$

In case of $\mathtt{Set}$, $A(X)$ is the quotient of the disjoint union of $A(n)$ by the equivalence relation $A(n)_f \ni x \sim A(\sigma)x \in A(n)_{\sigma \circ f}$:

$$A(X) = (\bigsqcup_{f : X \tilde{\to} [n]} A(n))/ \sim .$$

Before giving the definition of an operad in $\mathtt{Fun}(\mathcal{S}^{op}, \mathcal{C})$, we need to introduce some more notation and definitions. Define

$$Surj[Y, X] := \{h |\, X \stackrel{h}{\twoheadrightarrow} Y\} \text{ and } Surj[n, m] := Surj[[n], [m]]. \tag{1.5}$$

Definition 1.2.2. *For a surjection $f \in Surj[n, X]$ and A in $\mathcal{C}$ define*

$$A[f] = A(f^{-1}(1)) \odot \cdots \odot A(f^{-1}(n)).$$

Then define

$$A[n, X] := \coprod_{f \in Surj[n,X]} A[f].$$

Remark 1.2.3. We use the notation $[n, X]$ corresponding to $[n] \twoheadleftarrow X$ to emphasize that the Σ_n-action on $A[n, X]$ is a left action on the range $[n]$ and the $End_{\mathcal{S}}(X)$-action is a right action on the domain X. Another reason, which will become clearer after we give an alternative definition of an operad with respect to the category $\mathtt{Fun}(\widehat{\mathcal{S}}^{op}, \mathcal{C})$ is to have the composition $A[g] \odot A[f] \to A[g \circ f]$ for $g \in Surj[n, m]$ and $f \in Surj[m, l]$ take the same form as the composition law in the associativity axiom in Definition 1.2.1.

Recall that we are assuming that even if the monoidal structure is not strict, the associativity constraint for the monoidal structure relative to $\odot$ is simple enough that there is no need to specify brackets in the multiple product.

For any $\sigma \in \Sigma_n$ and $f \in Surj[n, X]$, there is a morphism $\hat{\sigma} : A[f] \to A[\sigma \circ f]$ defined by

$$\widehat{\sigma} : A(f^{-1}(1)) \odot \cdots \odot A(f^{-1}(n)) \to A\left(f^{-1}(\sigma^{-1}(1))\right) \odot \cdots \odot A\left(f^{-1}(\sigma^{-1}(n)\right) \tag{1.6}$$

given by the symmetric monoidal structure on the category $\mathcal{C}$.

Definition 1.2.4. *The left Σ_n-action on $A[n, X]$ is defined as the endomorphism of the coproduct $\coprod_{f \in Surj[n,X]} A[f]$ induced by the morphisms $\widehat{\sigma} : A[f] \to A[\sigma \circ f]$.*

Note that for any $g \in End_{\mathcal{S}}(X)$ there is a restricted bijection of finite sets

$$g : g^{-1}(f^{-1}(y)) \to f^{-1}(y) \text{ and by contravariance } A(g) : A(f^{-1}(y)) \to A(g^{-1}(f^{-1}(y))).$$

By functoriality of the $\odot$ product we have $A(g) : A[f] \to A[f \circ g]$.

Definition 1.2.5. *The right action of $g \in End_{\mathcal{S}}(X)$ on $A[n, X]$ is defined as the endomorphism of the coproduct $\coprod_{f \in Surj[n,X]} A[f] = A[n, X]$ induced by the morphisms $A(g) : A[f] \to A[f \circ g]$.*

For the special case $A[n, m] = \coprod_{f \in Surj[n,m]} A[f]$, there is a left Σ_n-action and a right Σ_m-action.

We need to extend the definition of $A[f]$ to surjections $f : X \twoheadrightarrow Y$ with arbitrary finite codomain Y, but with no preferred ordering. In order to do so we can apply a construction similar to the extension functor described in Proposition 1.2.13. Let $f : X \twoheadrightarrow Y$ be a surjection onto a finite set Y of cardinality n. An ordering of Y is, by definition, the same as a bijection $g : Y \xrightarrow{\sim} [n]$. For any such bijection, we have just defined $A[g \circ f]$. Now let

$$A'[f] := \coprod_{g : Y \xrightarrow{\sim} [n]} A[g \circ f].$$

Definition 1.2.6. *For each $\sigma \in \Sigma_n$, let $A'(\sigma) : A'[f] \to A'[f]$ be the morphism induced by the left action $\widehat{\sigma} : A[g \circ f] \to A[\sigma \circ g \circ f]$.*

Then in the case of an arbitrary finite set Y and $f \in Surj[Y, X]$ we define $A[f]$ to be the coequalizer of the morphisms $A'(\sigma)$ and

$$A[Y, X] := \coprod_{f \in Surj[Y,X]} A[f].$$

The left $End_{\mathcal{S}}(Y)$-action is given by extending the formula of Definition 1.2.4 for $\hat{\sigma}$. To formulate the equivariance axiom in the category $\mathtt{Fun}(\mathcal{S}^{op}, \mathcal{C})$ we need another definition.

For $A, B \in \mathtt{Fun}(\mathcal{S}^{op}, \mathcal{C})$, we use the diagonal $End_{\mathcal{S}}(Y)$-action to define

$$\begin{aligned} & A[Y] \odot_{End_{\mathcal{S}}(Y)} B[Y,X] \\ := \quad & \text{coequalizer}_{\ End_{\mathcal{S}}(Y)} \left(A[Y] \odot B[Y,X] \xrightarrow{\sigma^{-1} \odot \sigma} A[Y] \odot B[Y,X] \right). \end{aligned} \tag{1.7}$$

For $Y = [n]$ we use the more standard notation

$$A[n] \odot_{\Sigma_n} B[n,X] := A[Y] \odot_{End_{\mathcal{S}}([n])} B[n,X].$$

May's definition of an operad can now be reformulated in terms of finite sets and surjections.

Definition 1.2.7. *(Operad - Finite sets and surjections version) An* **operad** *is an object* $A \in \mathtt{Fun}(\mathcal{S}^{op}, \mathcal{C})$ *together with a family of morphisms* $\gamma_{Y,f} : A(Y) \odot A[f] \longrightarrow A(X)$ *(one for each triple consisting of a pair of finite sets* X, Y *and a surjection* $f : X \twoheadrightarrow Y$*) satisfying the axioms:*

1. *Associativity. Given three finite sets X,Y,Z and two surjections* $f : X \twoheadrightarrow Y$ *and* $g : Y \twoheadrightarrow Z$ *then the following diagram commutes*

$$\begin{tikzcd} A(Z) \odot A[g] \odot A[f] \arrow[r] \arrow[d, "\gamma_{Z,g} \odot id"'] & A(Z) \odot A[g \circ f] \arrow[d, "\gamma_{Z,g\circ f}"] \\ A(Y) \odot A[f] \arrow[r, "\gamma_{Y,f}"] & A(X) \end{tikzcd}$$

2. *Equivariance. The composition law expressed as a morphism* $A(Y) \odot A[Y,X] \to A(X)$ *factors through* $A(Y) \odot_{End_{\mathcal{S}}(Y)} A[Y,X]$.

3. *Unit. If* $\mathbb{1}$ *is the unit object of* $\mathcal{C}$*, then there is a morphism* $\eta : \mathbb{1} \to A(1)$ *such that for any finite set X with n elements, the composite morphisms*

$$\begin{aligned} id \odot \eta^{\odot n} &: A(X) \odot \mathbb{1}^{\odot n} \to A(X) \odot A[id_X] \to A(X) \\ \eta \odot id &: \mathbb{1} \odot A(X) \to A(1) \odot A(X) \longrightarrow A(X) \end{aligned}$$

are respectively the iterated left unit morphism and the right unit morphism for the underlying monoidal category $\mathcal{C}$. *(Here we use the natural identification of* $A(1)$ *and* $A(\{x\})$ *for any singleton set* $\{x\}$.*)*

If we restrict $A \in \mathtt{Fun}(\mathcal{S}^{op}, \mathcal{C})$ to the finite sets $[n]$, then these axioms reduce to the axioms in Definition 1.2.1. Conversely given $A \in \mathtt{Fun}(\widehat{\mathcal{S}}^{op}, \mathcal{C})$ satisfying the axioms of Definition 1.2.1 the extension functor gives an $EA \in \mathtt{Fun}(\mathcal{S}^{op}, \mathcal{C})$ satisfying the axioms of Definition 1.2.7.

Recalling our Remark 1.2.3 about the notation $A[n, X]$, we note that the sequence $Z \overset{g}{\twoheadleftarrow} Y \overset{f}{\twoheadleftarrow} X$ composes as $g, f \to g \circ f$, and the composition appearing in the associativity axiom

$$A[g] \odot A[f] \to A[g \circ f]$$

corresponds to the composition of the top arrows of the diagram in Definition 1.2.1 and the composition $\gamma_{Z,g} \odot id$ is the left down arrow. Using this definition, we see that operads form a subcategory $\mathtt{Op}_{\mathcal{C}}$ of $\mathtt{Fun}(\mathcal{S}^{op}, \mathcal{C})$. When $\mathcal{C}$ is understood, we denote it simply $\mathtt{Op}$.

Next we consider the definition of an operad in terms of a monoidal structure $\square$ on $\mathcal{D}$ which combines the data for all the composition operations.

Definition 1.2.8. *Define the $\square$ product on* $\mathtt{Fun}(\mathcal{S}^{op}, \mathcal{C})$ *by*

$$(A \square B)(X) := \coprod_{n=1}^{card(X)} (A[n] \odot_{\Sigma_n} B[n, X]).$$

Again for the special case $X = [m]$, we have

$$(A \square B)[m] = \coprod_{n=1}^{m} (A[n] \odot_{\Sigma_n} B[n, m]). \tag{1.8}$$

Steve: Jim asked me to put the following comment earlier. Where?

Smirnov [110] introduced the monoidal structure $\square$ on $\mathtt{Fun}(\widehat{\mathcal{S}}^{op}, \mathcal{C})$ in order to describe operads as monoids. Joyal had studied the same construction in the context of genral category theory under the name *species*, [54]. Category theorists call $\widehat{\mathcal{S}}$ the permutation category. A species is a functor from $\widehat{\mathcal{S}}$ to a category $\mathcal{C}$, which is symmetric monoidal, closed, complete and cocomplete. The category of species has a monoidal product called the substitution product which is essentially the $\square$ product, but is expressed in terms of partitions instead of surjections.

We verify that $\square$ satisfies the conditions for a monoidal structure.

Proposition 1.2.9. *If the monoidal category $\mathcal{C}$ has an initial object $\mathbf{0}$, then a unit object for $\square$ is given by the functor defined on objects by*

$$\mathbb{1}_{\mathtt{Fun}(\mathcal{S}^{op}, \mathcal{C})}(X) := \mathbb{1}_{\mathcal{C}} \text{ when } card(X) = 1, \text{ and} \tag{1.9}$$

$$\mathbb{1}_{\mathtt{Fun}(\mathcal{S}^{op}, \mathcal{C})}(X) := \mathbf{0} \text{ otherwise.} \tag{1.10}$$

The definition on morphisms follows from the properties of the unit and the initial object.

PROOF. Both identities

$$(\mathbb{1}\Box A)[n] \cong A[n] \quad \text{and}$$
$$(A\Box \mathbb{1})[n] \cong A[n]$$

follow immediately from the unitarity axioms and the identities

$$\mathbf{0} \odot X \cong \mathbf{0} \cong X \odot \mathbf{0} \text{ and} \tag{1.11}$$
$$\mathbf{0} \coprod X \cong X \cong X \coprod \mathbf{0}. \tag{1.12}$$

Since the initial object is the colimit over an empty index set, the identities in (1.11) follow from our assumption that the the multiplication $\odot$ on $\mathcal{C}$ commutes with colimits. The identities in (1.12) are implied by the Yoneda lemma and the following identities:

$$Hom_{\mathcal{C}}(X \coprod Y, Z) \cong Hom_{\mathcal{C}}(X, Z) \times Hom_{\mathcal{C}}(Y, Z) \text{ and } Hom_{\mathcal{C}}(\mathbf{0}, Z) \cong \{1\}. \qquad \blacksquare$$

Next we need to consider the associativity constraint for $\Box$ which obviously involves reshuffling various factors. A more precise description requires careful consideration of the orbits of the right and left actions of the symmetric group on the set of surjections. The crucial identity in proving the associativity of $\Box$ is given in the following proposition which will prove useful in later when we compare different characterizations of the Koszul property.

Proposition 1.2.10. *For each $A, B \in \mathtt{Fun}(\widehat{\mathcal{S}}^{op}, \mathcal{C})$ and natural numbers $n, p \geq 1$*

$$(A\Box B)[n,p] \cong \coprod_{n \leq m \leq p} A[n,m] \odot_{\Sigma_m} B[m,p].$$

In the case of $n = 1$ this reduces to the definition of $A\Box B$ on $\mathtt{Fun}(\widehat{\mathcal{S}}^{op}, \mathcal{C})$ The details of the proof are rather technical and will be deferred to the appendix to this section.

The associativity isomorphism for $\Box$ comes from the isomorphism in Proposition 1.2.10:

$$\begin{aligned}
((A\Box B)\Box C)[p] &= \coprod_{m\leq p} (A\Box B)[m] \odot_{\Sigma_m} C[m,p] \\
&\cong \coprod_{n\leq m\leq p} (A[n] \odot_{\Sigma_n} B[n,m]) \odot_{\Sigma_m} C[m,p] \\
&\cong \coprod_{n\leq m\leq p} A[n] \odot_{\Sigma_n} (B[n,m] \odot_{\Sigma_m} C[m,p]) \qquad (1.13) \\
&\cong \coprod_{n\leq p} A[n] \odot_{\Sigma_n} (B\Box C)[n,p] \\
&= (A\Box(B\Box C))[p].
\end{aligned}$$

Definition 1.2.11. *(Operad as monoid) An operad is a monoid for the $\Box$ product.*

Thus a $\mathbf{k}$-*linear operad* is a monoid in $\mathtt{Fun}(\mathcal{S}^{op}, \mathtt{Mod}_{\mathbf{k}})$, and an operad in $\mathtt{Set}$ is a monoid in $\mathtt{Fun}(\mathcal{S}^{op}, \mathtt{Set})$ for the monoidal structure given by their respective $\Box$-products.

Theorem 1.2.12. *Definition 1.2.1 (or Definition 1.2.7) is equivalent to Definition 1.2.11.*

Proof. Since $A\Box A$ is a coproduct of terms $A(n) \odot_{\Sigma_n} (A(m_1) \odot \cdots \odot A(m_n))$, a monoid structure $\mu : A\Box A \to A$ defines a set of operations $\gamma_{n,m_1,\ldots,m_n}$ of the type appearing in Definition 1.2.1. The equivariance axiom for these operations follows from the fact that the $\Box$ product is defined using the tensor product $\odot_{\Sigma_n}$, see equation (1.7). The commutative diagram expressing the associativity axiom in Definition 1.2.1 is equivalent to

$$\mu(\mu \odot id) = \mu(id \odot \mu) a_{A,A,A} : (A\Box A)\Box A \longrightarrow A.$$

This follows from the description of $a_{A,A,A}$ as the composition of the isomorphisms in (1.13) which have the effect of shuffling terms in the same way as the morphism ρ at the top of the commutative diagram (1.3). The unit axiom in May's definition is equivalent to the fact that $\mathbb{1}_{\mathtt{Fun}(\mathcal{S}^{op},\mathcal{C})}$ is a unit for the product μ. ■

1.2.1. Appendix to Section 1.2.

The extension EA of a functor A initially defined on the objects $[n]$ of $\hat{\mathcal{S}}$ to an arbitrary $X \in \mathcal{S}$ is defined as follows. If n is the cardinality of the finite set X, then let $A'(X)$ be the coproduct of $A(n)$ over an index set consisting of all bijections of X with $[n]$,

$$A'(X) := \coprod_{f:X \tilde{\to} [n]} A(n).$$

For each $\sigma \in \Sigma_n$ there is an endomorphism $A'(\sigma)_X : A'(X) \to A'(X)$ induced by the isomorphisms $A(\sigma^{-1}) : A(n)_f \to A(n)_{\sigma\circ f}$ between factors of the coproduct. Let $(EA)(X)$ be the coequalizer of the family of morphisms $A'(\sigma)_X$.

The universal property of coequalizers states, first, that there is a morphism $\pi_X : A'(X) \to EA(X)$ such that $\pi_X \circ A'(\sigma)_X = \pi_X$ for all $\sigma \in \Sigma_n$ and, further, for any object B and morphism $\psi : A'(X) \to B$ such that $\psi \circ A'(\sigma)_X = \psi$ for all $\sigma \in \Sigma_n$, there is a unique morphism $\tilde{\psi} : EA(X) \to B$ such that $\tilde{\psi} \circ \pi_X = \psi$. Using this property we can define $EA(g)$ for a morphism $g : X \to Y$. First, there is a morphism $A'(g) : A'(Y) \to A'(X)$ induced by the map of index sets sending $f : Y \tilde{\to} [n]$ to $f \circ g : X \tilde{\to} [n]$. Associativity $\sigma \circ (f \circ g) = (\sigma \circ f) \circ g$ implies $A'(g) \circ A'(\sigma)_Y = A'(\sigma)_X \circ A'(g)$. Composing with π_X we have $\pi_X \circ A'(g) \circ A'(\sigma)_Y = \pi_X \circ A'(g)$. Then the universal property of $EA(Y)$ implies that there is a morphism, $EA(g) : EA(Y) \to EA(X)$, such that $EA(g) \circ \pi_Y = \pi_X \circ A'(g)$.

Proposition 1.2.13. *The functors* $R : \mathtt{Fun}(\mathcal{S}^{op}, \mathcal{C}) \to \mathtt{Fun}(\widehat{\mathcal{S}}^{op}, \mathcal{C})$ *and* $E : \mathtt{Fun}(\widehat{\mathcal{S}}^{op}, \mathcal{C}) \to \mathtt{Fun}(\mathcal{S}^{op}, \mathcal{C})$ *are inverse functors.*

Proof. First we show that for $A \in \mathtt{Fun}(\widehat{\mathcal{S}}^{op}, \mathcal{C})$, $EA(n)$ is naturally isomorphic to $A(n)$ for all natural numbers n. By definition $EA(n)$ is a coequalizer of the family of endomorphisms $A'(\sigma)$ acting on $A'(n) = \coprod_{\sigma \in \Sigma_n} A(n)$. There is a morphism $\phi : A'(n) \to A(n)$ given on the component $A(n)_\sigma$ by $A(\sigma)$ which satisfies $\phi \circ A'(\sigma)_{[n]} = \phi$. Furthermore, given $\psi : A'(n) \to B$ as above, if we define $\tilde{\psi} = \psi \circ j$ where $j : A(n) \to A'(n)$ is the inclusion $A(n)$ at index $\sigma = id$, then $\tilde{\psi} \circ \phi = \psi$. Clearly $\tilde{\psi}$ is unique. Thus $A(n)$ has the properties of a coequalizer and by uniqueness is naturally isomorphic to $EA(n)$. This shows that $RE \cong id_{\mathtt{Fun}(\widehat{\mathcal{S}}^{op}, \mathcal{C})}$. To show that for $A \in \mathtt{Fun}(\mathcal{S}^{op}, \mathcal{C})$, $ER \cong id_{\mathtt{Fun}(\mathcal{S}^{op}, \mathcal{C})}$, we need to prove that there is a natural isomorphism $EA(X) \cong A(X)$, for $A \in \mathcal{D}$. In this case we note that if A is defined on $\mathcal{S}$, then for any bijection, $f : X \to [n]$, we have a morphism $A(f) : A(n)_f \to A(X)$. By the universal property of the coproduct, this defines a morphism $\phi_X : A'(X) \to A(X)$.

By contravariance, $A(\sigma \circ f) = A(f) \circ A(\sigma)$, or $A(\sigma \circ f) \circ A(\sigma)^{-1} = A(f)$, implying $\phi_X \circ A'_X(\sigma) = \phi_X$. Thus there is a morphism $\alpha : EA(X) \to A(X)$. Picking one bijection $g : [n] \to X$ we get a morphism $A(g) : A(X) \to A'(X)$ and $\beta = \pi_X \circ A(g)$ is inverse to α. ∎

Proof of Proposition 1.2.10. By definition, $(A \square B)[n,p] = \coprod_{f \in Surj[n,p]} (A \square B)[f]$. A given surjection $f : [p] \twoheadrightarrow [n]$ partitions $[p]$ into n subsets, and then $A \square B$ runs over all partitions of each of these subsets. By definition

$$\begin{aligned}
(A \square B)[f] &= \bigodot_{1 \le i \le n} (A \square B)(f^{-1}(i)) \\
&= \bigodot_{1 \le i \le n} \left(\coprod_{m_i = 1}^{card(f^{-1}(i))} A(m_i) \odot_{\Sigma_{m_i}} \left[\coprod_{g_i \in Surj[[m_i], f^{-1}(i)]} B[g_i] \right] \right) \\
&= \bigodot_{1 \le i \le} \left(\coprod_{m_i = 1}^{card(f^{-1}(i))} A(m_i) \odot_{\Sigma_{m_i}} \left[\coprod_{g_i \in Surj[[m_i], f^{-1}(i)]} \left\{ \bigodot_{1 \le j \le m_i} B(g_i^{-1}(j)) \right\} \right] \right).
\end{aligned}$$

The rather complicated indexing describes what one might call "a nested partition", the $f^{-1}(i)$ partition $[p]$ into n subsets and the $g_i^{-1}(j)$ for $1 \le j \le m_i$ partition each $f^{-1}(i)$ into m_i subsets. We can distribute the first $\odot$ product (indexed by i) over the adjacent coproduct and permute factors so that all the $A(m_i)$ terms are together and all the $B(g_i^{-1}(j))$ terms are together. When $(A \square B)[f]$ is rewritten in this form, we have a coproduct over all nested partitions of $[p]$ given by the $f^{-1}(i)$ and $g_i^{-1}(j)$.

$$(A \square B)[f] \cong \coprod_{\substack{\mathbf{m} \in \mathbf{N}^n \\ 1 \le m_i \le \#(f^{-1}(i))}} \left(\bigodot_{1 \le i \le n} A(m_i) \right) \odot_{\Sigma_{m_1} \times \cdots \times \Sigma_{m_n}} \left(\coprod_{g = \sqcup g_i} \left[\bigodot_{j \in \sqcup [m_i]} B(g^{-1}(j)) \right] \right).$$

Let $m := \sum_{i=1}^{n} m_i$ whenever $\mathbf{m} = (m_1, \cdots, m_n) \in \mathbf{N}^n$ and define a "standard partition" via the non-decreasing surjection $h_{\mathbf{m}} : [m] \twoheadrightarrow [n]$ by

$$h_{\mathbf{m}}(j) = i \text{ for } m_1 + .. + m_{i-1} < j \leq m_1 + .. + m_i. \tag{1.14}$$

Then we can write the preceding sum as:

$$\coprod_{\{\mathbf{m}|\ 1\leq m_i \leq \#(f^{-1}(i))\}} A[h_{\mathbf{m}}] \odot_{\Sigma_{m_1}\times\cdots\times\Sigma_{m_n}} \left(\coprod_{\substack{g\in Surj[m,p] \\ h\circ g = f}} B[g] \right) \cong$$

$$\coprod_{\{\mathbf{m}|\ 1\leq m_i \leq \#(f^{-1}(i))\}\}} (A[h_{\mathbf{m}}] \odot_{\Sigma_{m_1}\times\cdots\times\Sigma_{m_n}} \mathbb{1}[m,m]) \odot_{\Sigma_m} \left(\coprod_{\substack{g\in Surj[m,p] \\ h\circ g = f}} B[g] \right).$$

If we now take the direct sum over the surjections $f : [p] \twoheadrightarrow [n]$, this removes the conditions on the individual m_i and allows for arbitrary g, so that $A[n,p] = \coprod_{f\in Surj[n,p]} (A\square B)[f]$ is isomorphic to:

$$\coprod_{\substack{\mathbf{m}\in\mathbf{N}^n \\ m\leq p}} A[h_{\mathbf{m}}] \odot_{\Sigma_{m_1}\times\cdots\times\Sigma_{m_n}} \mathbb{1}[m,m]) \odot_{\Sigma_m} \left(\coprod_{g\in Surj[m,p]} B[g] \right). \tag{1.15}$$

To complete the proof, we need to show that this is isomorphic to $\coprod_{n\leq m\leq p} A[n,m] \odot_{\Sigma_m} B[m,p]$, and since the factor on the right in (1.15) equals $B[m,p]$, it is enough to show (if we ignore for the time being the left Σ_n module structure):

Lemma 1.2.14. *There is a right Σ_m isomorphism:*

$$A[n,m] \cong \coprod_{\{\mathbf{m}|\sum m_i = m\}} (A[h_{\mathbf{m}}] \odot_{\Sigma_{m_1}\times\cdots\times\Sigma_{m_n}} \mathbb{1}[m,m]).$$

The index set for the coproduct is the set of ordered partitions of m into n nonzero parts.

Proof. Define $\gamma : Surj[n,m] \to \mathbf{N}^n$ by $\gamma(h) = (m_1, \cdots, m_n)$ where $m_i = card(h^{-1}(i))$. Then γ is constant along the orbits of the right action of Σ_m on $Surj[n,m]$ and separates orbits, so the induced map $\overline{\gamma} : Surj[n,m]/\Sigma_m \to \mathbf{N}^n$ gives coordinates on the orbit space. Take the surjection $h_{\mathbf{m}}$ (as in 1.14) to be the base point of the orbit $h_{\mathbf{m}} \circ \Sigma_m$; then the isotropy group is $\Sigma_{m_1} \times \cdots \times \Sigma_{m_n}$, and there is a bijection

$$h_{\mathbf{m}} \circ \Sigma_m \longleftrightarrow (\Sigma_{m_1} \times \cdots \times \Sigma_{m_n})\backslash\Sigma_m.$$

Hence

$$Surj[n,m] = \bigsqcup_{\{\mathbf{m}\in\mathbf{N}^n \,|\, \sum m_i = m\}} h_{\mathbf{m}} \circ \Sigma_m \cong \bigsqcup_{\{\mathbf{m}\in\mathbf{N}^n \,|\, \sum m_i = m\}} (\Sigma_{m_1} \times \cdots \times \Sigma_{m_n})\backslash\Sigma_m.$$

If we define the subobject $A\langle\mathbf{m}\rangle$ of $A[n,m]$ by

$$A\langle\mathbf{m}\rangle := \coprod_{f\in h_{\mathbf{m}}\circ\Sigma_m} A[f],$$

then $A\langle\mathbf{m}\rangle$ has an obvious right Σ_m structure, and the right Σ_m decomposition (into irreducible subobjects) of $A[n,m]$ is

$$A[n,m] = \coprod_{f:[m]\to[n]} A[f] = \coprod_{\{\mathbf{m}\in\mathbf{N}^n \mid \sum m_i = m\}} A\langle\mathbf{m}\rangle. \tag{1.16}$$

Each $A\langle\mathbf{m}\rangle$ is isomorphic to $A[h_{\mathbf{m}}]\odot_{\Sigma_{m_1}\times\cdots\times\Sigma_{m_n}} \mathbb{1}[m,m]$ by the map $\alpha\odot\sigma \longmapsto \alpha\cdot\sigma$ which is defined by the right $End_{\mathcal{S}}([n])$-action on $A[n,m]$ discussed above. This clearly defines a right Σ_m module isomorphism, concluding the proof of the lemma. ■

Returning now to the left Σ_n module structure on $A[n,m]$, we need another definition.

Definition 1.2.15. *Let $(m_i) := (m_1,\cdots,m_n)$ be an ordered partition, $m = \sum m_i$, and $\sigma \in \Sigma_n$, we define $\varphi_{(m_1,\cdots,m_n)}(\sigma) \in \Sigma_m$ acting on the set $[m]$ by permuting n intervals of lengths $m_1, m_2, \ldots, m_n$. More precisely, define a new partition, $(m'_i) := (m'_1,\ldots,m'_n)$ where $m'_i = m_{\sigma^{-1}(i)}$. Then $\varphi_{(m_1,\cdots,m_n)}(\sigma)$ is one to one monotonic from the i-th interval $\{j|m_1+..+m_{i-1} < j \le m_1+..+m_i\}$ of the partition (m_i) onto the $\sigma(i)$-th interval $\{k|m'_1+..+m'_{\sigma(i)-1} < k \le m'_1+..+m'_{\sigma(i)}\}$ of the partition (m'_i).*

Example 1.2.16. $n=3$, $m=6$,

$$\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix},$$

and $(m_1,m_2,m_3) = (2,1,3)$. Then $(m'_1,m'_2,m'_3) = (1,3,2)$, the "blocks" determined by m and m' are respectively $(12 \mid 3 \mid 456)$ and $(1 \mid 234 \mid 56)$, and

$$\varphi_{(2,1,3)}(\sigma) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 6 & 1 & 2 & 3 & 4 \end{pmatrix}.$$

Clearly $\sigma\circ h_{(m_1,\cdots,m_n)}\circ\varphi_{(m_1,\cdots,m_n)}(\sigma)^{-1} = h_{(m_{\sigma^{-1}(1)},\cdots,m_{\sigma^{-1}(n)})}$. For any $\sigma\in\Sigma_n$ we define a right Σ_m isomorphism

$$\widehat{\sigma} : A[h_{(m_1,\cdots,m_n)}]\odot_{\Sigma_{m_1}\times\cdots\times\Sigma_{m_n}} \mathbb{1}[m,m] \to A[h_{(m_{\sigma^{-1}(1)},\cdots,m_{\sigma^{-1}(n)})}]\odot_{\Sigma_{m_{\sigma^{-1}(1)}}\times\cdots\times\Sigma_{m_{\sigma^{-1}(n)}}} \mathbb{1}[m,m]$$

by $\widehat{\sigma}(\alpha\odot\tau) = \sigma\cdot\alpha\cdot\varphi_{(m_1,\cdots,m_n)}(\sigma)^{-1}\odot\left[\varphi_{(m_1,\cdots,m_n)}(\sigma)\right]\tau$. This is well defined, since for $\rho\in\Sigma_{m_1}\times\cdots\times\Sigma_{m_n}$, and $\varphi_{(m_1,\cdots,m_n)}(\sigma)\rho\varphi_{(m_1,\cdots,m_n)}(\sigma)^{-1}\in\Sigma_{m_{\sigma^{-1}(1)}}\times\cdots\times\Sigma_{m_{\sigma^{-1}(n)}}$, we have

$$\begin{aligned}
\widehat{\sigma}([\alpha\cdot\rho]\odot\tau) &= \sigma\cdot[\alpha\cdot\rho]\cdot\varphi_{(m_1,\cdots,m_n)}(\sigma)^{-1}\odot\varphi_{(m_1,\cdots,m_n)}(\sigma)\tau \\
&= (\sigma\cdot\alpha\cdot\varphi_{(m_1,\cdots,m_n)}(\sigma)^{-1})\cdot(\varphi_{(m_1,\cdots,m_n)}(\sigma)\rho\varphi_{(m_1,\cdots,m_n)}(\sigma)^{-1})\odot\varphi_{(m_1,\cdots,m_n)}(\sigma)\tau \\
&= \sigma\cdot\alpha\cdot\varphi_{(m_1,\cdots,m_n)}(\sigma)^{-1}\odot(\varphi_{(m_1,\cdots,m_n)}(\sigma)\rho\varphi_{(m_1,\cdots,m_n)}(\sigma)^{-1})\varphi_{(m_1,\cdots,m_n)}(\sigma)\tau \\
&= \widehat{\sigma}(\alpha\odot\rho\tau).
\end{aligned}$$

For each non-increasing partition of m into n parts, $m_1 \geq m_2 \geq \ldots \geq m_n$, define

$$A\{\mathbf{m}\} = A\{m_1, \cdots, m_n\} := \coprod_{\sigma \in \Sigma_n}{}' A[h_{(m_{\sigma(1)}, \cdots, m_{\sigma(n)})}] \odot_{\Sigma_{m_{\sigma(1)}} \times \cdots \times \Sigma_{m_{\sigma(n)}}} \mathbb{1}[m, m],$$

where the $'$ indicates that the coproduct ranges over the distinct ordered partitions given by permuting the orginal non-increasing partition.

The orbits of the left Σ_n-action on $A[n, m]$ will be denoted by

$$\overline{A\{\mathbf{m}\}} = \overline{A\{m_1, \cdots, m_n\}} := \coprod_{\sigma \in \Sigma_n}{}' A\langle m_{\sigma(1)}, \cdots, m_{\sigma(n)}\rangle.$$

Lemma 1.2.17. *The composition map $\alpha \odot \tau \longmapsto \alpha \cdot \tau$ defines a right Σ_m left Σ_n isomorphism of $A\{m_1, \cdots, m_n\}$ and $\overline{A\{m_1, \cdots, m_n\}}$.*

The proof of the lemma is clear from the preceding discussion and completes the proof of Proposition 1.2.10. ∎

1.3. Pseudo-operads.

We find it useful to use yet another structure inherent in operads, the "$\circ_i$ - products" introduced by Gerstenhaber [6] in his study of the graded Lie algebra structure on Hochschild cohomology of an associative algebra. For an operad A, and natural numbers m, n and some j, $1 \leq j \leq m$, one can define the map $\circ_j$ (which also depends on m and n) as a composition:

$$\circ_i : A(m) \odot A(n) \overset{\cong}{\to} A(m) \odot (A(1) \odot \cdots \odot A(n) \odot \cdots \odot A(1)) \xrightarrow{\gamma_{m,1,\ldots,n,\ldots,1}} A(m+n-1) \quad (1.17)$$

where $A(n)$ is placed in position i of the factor in parentheses. The structure maps $\gamma_{j, r_1, \cdots, r_j}$ for arbitrary r_i can be recovered by using suitable compositions of the $\circ_i$-product maps. Also, using this approach one can do away with the multi-indexing inherent in the standard approach which uses γ structure maps.

These operations were used by Markl [16] to define "pseudo-operads" or "non-unitary" operads, which satisfy associativity and equivariance axioms but do not assume the existence of a unit in $A(1)$. In the definition below operation $f \circ_i g$ stands for "substitution of g in position i of f" which is the reverse of the convention adopted in [16], accordingly, the axioms below differ from those in [16].

To deal with permutations we need to define a somewhat unusual composition law for permutations. Let $\sigma \in \Sigma_m$, $\rho \in \Sigma_n$, and define $\sigma \circ_i \rho \in \Sigma_{m+n-1}$ by having ρ act on the interval $[i, i+n-1]$ followed by mapping the interval monotonically to the interval $[\sigma(i), \sigma(i)+n-1]$, permuting the remaining elements as in σ. This is $\varphi_{1, \cdots 1, n, 1 \cdots, 1}(\sigma)$ as described in the appendix to Section 1.2, Definition 1.2.15, composed with $(1 \times \cdots \times \rho \times \cdots \times 1)$,

$$\sigma \circ_i \rho := \varphi_{1, \cdots 1, n, 1 \cdots, 1}(\sigma)(1 \times \cdots \times \rho \times \cdots \times 1) \quad (1.18)$$

Definition 1.3.1. *(Markl's pseudo-operads) A pseudo-operad in a monoidal category $\mathcal{C}$ is an object $A \in \mathtt{Fun}(\widehat{\mathcal{S}}^{op}, \mathcal{C})$ together with composition operations*

$$\circ_i : A(a) \odot A(b) \longrightarrow A(a+b-1),$$

with the following "associativity" conditions for the iterated compositions of $A(a) \odot A(b) \odot A(c)$:

$$\circ_i (\circ_j \odot id) = \begin{cases} \circ_{j+c-1}(\circ_i \odot id)(id \odot \tau) & 1 \leq i \leq j-1 \\ \circ_j(id \odot \circ_{i-j+1}) & j \leq i \leq j+b-1 \\ \circ_j(\circ_{i-b+1} \odot id)(id \odot \tau) & j+b \leq i, \end{cases} \tag{1.19}$$

where $\tau : A(b) \odot A(c) \to A(c) \odot A(b)$ is the transposition given by the symmetry.

The operation $\circ_i$ is equivariant in the sense:

$$\circ_i (\sigma \odot \rho) = (\sigma \circ_i \rho) \circ_{\sigma(i)} \text{ on } A(m) \odot A(n). \tag{1.20}$$

where $(\sigma \circ_i \rho)$ is defined in equation (1.18).

Using the standard procedure, we can extend the pseudo-operad structure on $A \in \hat{D}$ to $E(A) \in \mathtt{Fun}(\mathcal{S}^{op}, \mathcal{C})$ by defining, for $x \in X$, composition operations

$$\circ_x : A(X) \odot A(Y) \longrightarrow A(X \sqcup_x Y),$$

where

$$X \sqcup_x Y := (X - x) \cup Y. \tag{1.21}$$

The definition is straightforward. Given $f : [n] \xrightarrow{\sim} X$ and $g : [m] \xrightarrow{\sim} Y$, with $f(i) = x$ define $f \sqcup_x g : [m+n-1] \to X \sqcup_x Y$, by

$$(f \sqcup_x g)(j) = \begin{cases} f(j),\ 1 \leq j \leq i-1, \\ g(j-i+1),\ i \leq j \leq i+m-1, \\ f(j-m+1),\ i+m \leq j \leq m+n-1. \end{cases}$$

Then define $\circ_x$ as the composition

$$A(X) \odot A(Y) \xrightarrow{A(f)\odot A(g)} A(n) \odot A(m) \xrightarrow{\circ_i} A(n+m-1) \xrightarrow{A((f \sqcup_x g)^{-1})} A(X \sqcup_x Y). \tag{1.22}$$

The (pseudo-operad) equivariance axiom imply that $\circ_x$ is well-defined independent of the choice of f, g. Using the equalities of sets,

$$X \sqcup_x (Y \sqcup_y Z) = (X \sqcup_x Y) \sqcup_y Z \text{ and } (X \sqcup_{x_1} Y) \sqcup_{x_2} Z = (X \sqcup_{x_2} Z) \sqcup_{x_1} Y,$$

the associativity axioms become

$$\circ_x(id \odot \circ_y) = \circ_y(\circ_x \odot id) : A(X) \odot A(Y) \odot A(Z) \to A(X \sqcup_x (Y \sqcup_y Z)) \tag{1.23}$$
$$\circ_{x_2}(\circ_{x_1} \odot id) = \circ_{x_1}(\circ_{x_2} \odot id)(id \odot \tau) : A(X) \odot A(Y) \odot A(Z) \longrightarrow A((X \sqcup_{x_1} Y) \sqcup_{x_2} Z)$$

where $x, x_1, x_2 \in X$ and $y \in Y$.

The category of pseudo-operads in $\mathcal{C}$ will be denoted $\Psi\mathtt{Op}$.

Remark 1.3.2. The previous discussion shows that a pseudo-operad structure on an object $A \in \mathtt{Fun}(\mathcal{S}^{op}, \mathcal{C})$ is equivalent to the existence of $\circ_x$ operations defined on the unordered $\odot$ product in the symmetric monoidal category $\mathcal{C}$ together with the requirement that these operations satisfy the associativity equations (1.23). The $\circ_x$ operations are related in a very natural way to the existence of a functor from the category of trees to the category $\mathcal{C}$, which we describe in the next section. We will see that the language of trees is really more appropriate to pseudo-operads than operads, although it can be used in both contexts.

The relation between pseudo-operads and operads is made more precise in the next theorem, which shows that if A is a pseudo-operad, then it is possible to define operations $\gamma_{X,f} : A(X) \odot A[f] \to A(Y)$ which satisfy the first two axioms in Definition 1.2.7, and given a unit, we can define the full structure of an operad. Thus one could say an operad is a pseudo-operad with unit, however one must be careful here since it is NOT true that a pseudo-operad is an operad without a unit. The unit axiom for an operad is essential in order to define a pseudo-operad structure, that is, to derive the $\circ_i$ operations from the composition law γ.

Theorem 1.3.3. *If $A \in \mathtt{Fun}(\mathcal{S}^{op}, \mathcal{C})$ is a pseudo-operad then for $f : Y \twoheadrightarrow X$ there exist operations $\gamma_{X,f} : A(X) \odot A[f] \longrightarrow A(Y)$ satisfying the first two axioms of Definition 1.2.7. These operations are given by a composition, which although defined using an order on X, is independent of the order chosen:*

$$\gamma_{X,f} : A(X) \odot A(f^{-1}(x_1)) \odot \cdots \odot A(f^{-1}(x_n)) \xrightarrow{\circ_{x_n}(\circ_{x_{n-1}} \odot id) \cdots (\circ_{x_1} \odot \cdots \odot id)} A(Y). \tag{1.24}$$

The third axiom is fulfilled if there exist "unit" morphisms

$$\eta : \mathbb{1} \to A(1) \tag{1.25}$$

such that, when extended to $A(\{x\})$, for all finite sets $X \ni x$, the following diagrams commute

$$\begin{tikzcd} A(X) \odot \mathbb{1} \arrow[r, "r"] \arrow[d, "id \odot \eta_x"'] & A(X) \\ A(X) \odot A(\{x\}) \arrow[ur, "\circ_x"'] & \end{tikzcd} \qquad \begin{tikzcd} \mathbb{1} \odot A(X) \arrow[r, "l"] \arrow[d, "\eta_x \odot id"'] & A(X) \\ A(\{x\}) \odot A(X) \arrow[ur, "\circ_x"'] & \end{tikzcd} \tag{1.26}$$

Conversely, let A be an operad, then we can define $\circ_x$ operations on A giving it the structure of pseudo-operad as follows. Given two finite sets X, Y, first define a surjection

$$f : X \sqcup_x Y \twoheadrightarrow X \ , \ f(y) = x \text{ for } y \in Y, \text{ and } f(x') = x' \text{ for } x \neq x' \in X. \tag{1.27}$$

Let $\eta : \mathbb{1} \to A(\{x\})$ be the unit for the operad structure, and define $j : A(Y) \to A[f]$ by

$$A(Y) \xrightarrow{\cong} A(Y) \odot \mathbb{1}^{\odot n} \xrightarrow{id \odot \eta^{\odot n}} A[f].$$

Then define $\circ_x$ by the following diagram

$$\begin{tikzcd} A(X) \odot A(Y) \arrow[r, "\circ_x"] & A(X \sqcup_x Y) \\ A(X) \odot A[f] \arrow[u, "id \odot j"] \arrow[ur, "\gamma_{X,f}"'] & \end{tikzcd} \tag{1.28}$$

Proof. If A is a pseudo-operad, then the second equation in (1.23) implies that $\gamma_{X,f}$ is well-defined independent of the ordering of the x_i. This equation also implies the equivariance axiom of Definition 1.2.7. The associativity axiom in that definition follows from the pair of equations in (1.23). If there exists a unit as in equation (1.25) the third axiom for an operad structure is satisfied.

Conversely, if we are given an operad A and we define $\circ_x$ as in the statement of the theorem, then the associativity and equivariance axioms of Definition 1.2.7 imply equations (1.23). ∎

1.4. Free operads.

In this section we give two constructions of the "free operad" generated by an object in $\mathtt{Fun}(\mathcal{S}^{op}, \mathcal{C})$. The most familiar examples of free objects arise in the context of monoids in a monoidal category. Let $\mathcal{A}$ be a monoidal category with multiplication $\odot$. If the forgetful functor $\mathcal{F}_{or} : \mathtt{Mon}(\mathcal{A}) \to \mathcal{A}$ has a left adjoint, we denote it by $\mathcal{F}^{\odot}_{ree}$, and call $\mathcal{F}^{\odot}_{ree}(X)$ the free monoid with unit generated by $X \in \mathcal{A}$. For our standard examples

$$\mathcal{F}^{\otimes}_{ree}(X) = \bigoplus_{n \geq 0} X^{\otimes n} \text{ for } X \in \mathrm{Mod}_{\mathbf{k}} \text{ and } \mathcal{F}^{\times}_{ree}(X) = \bigsqcup_{n \geq 0} X^{\times n} \text{ for } X \in \mathtt{Set}.$$

Barr [4] noted that if $\mathcal{A}$ has countable coproducts and the multiplication commutes with coproducts in both arguments, then a general definition of the free monoid on $X \in \mathcal{A}$ is given by

$$\mathcal{F}^{\odot}_{ree}(X) = \coprod_{n \geq 0} X^{\odot n}.$$

Using this construction he defined a categorical bar complex and an associated cohomology theory.

For the Smirnov monoidal structure defined in the previous section, the multiplication $\Box$ is distributive over coproducts on the left but not on the right. For example, $(E\Box(F\coprod G))(2)$ contains a component $E(2)\odot F(1)\odot G(1)$ not appearing in $(E\Box F)(2)$ or $(E\Box G)(2)$ and thus in general

$$E\Box(F\coprod G) \neq (E\Box F)\coprod(E\Box G).$$

This causes a problem in Barr's definition of the free object since one cannot simply take the coproduct of the iterated powers. A more general construction has been described by Baues, Jilbadze, and Tonks in [3] and is valid for multiplications which are left distributive over coproducts and commute with filtered colimits on the left as in the case of the $\Box$ product. In Section 3.3 we use a slight modification of their construction to define the free monoid functor for $\Box$.

Coming back to the model $\mathcal{F}^{\odot}_{ree}(X) = \coprod_{n\geq 0} X^{\odot n}$ for the free monoid, we see that there is a subobject $X^{\odot n}$ corresponding to the input data for the n-fold iterated composition law for a monoid structure on X. By analogy we expect the free operad generated by $A \in \mathtt{Fun}(\mathcal{S}^{op}, \mathcal{C})$ to contain a subobject corresponding to the input data for each possible sequence of operadic compositions. This data is described by the combinatorial structure of a tree and consequently there is an alternative, very intuitive, construction of the free operad using trees. We discuss that approach first, after which we present the monoidal approach and then compare the two.

Recall some basic definitions.

Definition 1.4.1. *A tree is a finite connected contractible graph.*

According to the usual conventions all edges in a graph have two adjacent vertices. We modify this convention and delete the vertices with only one adjacent edge, so some edges will have only one adjacent vertex. We call these external edges. The edges which are adjacent to two vertices are called internal edges. A *rooted tree* has a distinguished external edge, called the *root.* The remaining external edges are called *leaves.* All trees are assumed to have at least one edge, the tree with just one edge and no vertices is called the *trivial tree.* A rooted tree has a natural orientation with each edge oriented in the direction of the (unique) path from that edge to the root, that is, the edge is oriented toward the vertex closer to the root. Then every vertex is adjacent to a single outgoing edge. The *arity* of the vertex is the number of incoming edges. Let $In(v)$ be the set of incoming edges at the vertex v. A tree all of whose vertices have arity at least two is called *reduced.* A planar tree is a tree together with an imbedding in the plane.

Trees form a full subcategory of the category of graphs. Morphisms in the category of graphs are maps of graphs sending vertices to vertices and edges to edges or vertices. (Since a map of graphs must preserve adjacency relations if an edge is mapped to a vertex, the adjacent vertices are mapped to the same vertex.) An *elementary morphism* $\pi_e : T \to T/e$

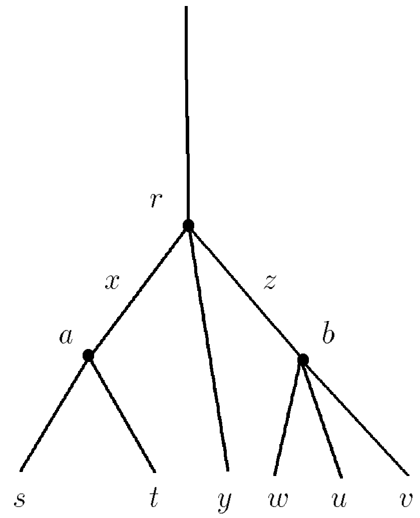


Figure 1.4: An up-rooted tree with root vertex r. Setting $f(1) = r, f(2) = a, f(3) = b$ gives $\bigodot_f A(In(v)) = A(\{x, y, z\}) \odot A(\{s, t\}) \odot A(\{u, v, w\})$.

contracts one internal edge e and leaves the complement of e unchanged. Every morphism is a composite of elementary morphisms and isomorphisms. We use the notation $\pi_{\{e_1,\ldots,e_n\}} : T \to T/\{e_1, \ldots, e_n\}$ to denote the composition of elementary morphisms cotracting the edges $\{e_1, \ldots, e_n\}$. Rooted trees form a subcategory in which morphisms preserve the root. Unless otherwise indicated, we will assume all trees are rooted and oriented in the above sense and denote the category of such trees by `Tree`. A *labeled tree* has labels assigned to each of the leaves. For such trees there are no nontrivial automorphisms that preserve labels of leafs, that is, the chains of elementary contractions are indeed the only morphisms of trees as graphs.

As mentioned earlier, we expect the free operad generated by $A \in \mathtt{Fun}(\mathcal{S}^{op}, \mathcal{C})$ to contain a subobject corresponding to the input data for each possible sequence of operadic compositions. Since a tree gives us a schematic description of a sequence of compositions of n-ary operations, if we can define an $\odot$ product of the objects $A(In(v))$ as v runs over the vertices of $T \in \mathtt{Tree}$, we will be able to represent the input data for this sequence of operadic compositions. To make this product depend only on the tree and not on the ordering of the vertices we have to define an $\odot$ product of objects A_x indexed by an arbitrary finite set X. The method we use is essentially the extension of a functor from natural numbers to finite sets which we saw in the previous section.

Given a set X of cardinality n, let $\mathrm{Ord(X)} = \{f | f : [n] \xrightarrow{\sim} X\}$ be the set of bijections, or equivalently, orderings of X. For each $f \in \mathrm{Ord(X)}$ define $\bigodot_f A_x := \bigodot_{i=1}^n A_{f(i)}$; see Figure 1.4. Using the symmetric structure on $\mathcal{C}$ we can define a permutation morphism $\overline{\sigma} : \bigodot_f A_X \to \bigodot_{f\circ(\sigma^{-1})} A_X$ for each $\sigma \in \Sigma_n$,

$$\overline{\sigma} : \bigodot{}_f A_x = A_{f(1)} \odot \cdots \odot A_{f(n)} \longrightarrow A_{f(\sigma^{-1}(1))} \odot \cdots \odot A_{f(\sigma^{-1}(n))} = \bigodot{}_{f\circ(\sigma^{-1})} A_x.$$

Then we extend $\bar{\sigma}$ to the coproduct over Ord(X) and define the tensor product over an arbitrary finite set.

Definition 1.4.2. *For an arbitrary finite set X, define*

$$\bigodot_X A_x := \underset{\sigma \in \Sigma_n}{\text{coequalizer}} \left\{ \overline{\sigma} : \coprod_{f \in \mathrm{Ord(X)}} \bigodot_f A_x \longrightarrow \coprod_{f \in \mathrm{Ord(X)}} \bigodot_f A_x \right\}.$$

A bijection g of the finite set X acts on the set Ord(X) by composition $f \mapsto g \circ f$ and defines a corresponding symmetry of the $\odot$-products,

$$\overline{g} : \bigodot_f A_x \to \bigodot_{g \circ f} A_x.$$

By the standard argument using the universal property of coequalizers, this defines a left-action of the group $End_{\mathcal{S}}(X)$ of bijections of X on $\bigodot_X A_x$. For any $A \in \mathtt{Fun}(\mathcal{S}^{op}, \mathcal{C})$ and finite set X there is also a left-action of $End_{\mathcal{S}}(X)$ on $A(X)$ given by composing the right-action (see Definition 1.2.7) with the inversion in the group. Then for any collection of finite sets indexed by X, $\{Y_x | \, x \in X\}$, we can define a diagonal left-action of $End_{\mathcal{S}}(X)$ on $A(X) \odot \bigodot_X A(Y_x)$.

Definition 1.4.3. *For a tree T with non-empty set of vertices $Vert(T)$, define $A(T) \in \mathcal{C}$,*

$$A(T) := \underset{\left\{ \substack{v \in Vert(T) \\ g \in End_{\mathcal{S}}(In(v))} \right\}}{\text{coequalizer}} \left\{ \overline{g} : \bigodot_{Vert(T)} A(In(v)) \longrightarrow \bigodot_{Vert(T)} A(In(v)) \right\}, \quad (1.29)$$

where $\overline{g}$ is the morphism defined by the left-action of $g \in End_{\mathcal{S}}(In(v))$.

For an operad or pseudo-operad A it is possible to extend this construction to define a functor from `Tree` to $\mathcal{C}$ by extending the definition of A to the morphisms in `Tree`. We will see that the the construction uses only the pseudo-operad structure.

Consider first an operad in $A \in \mathtt{Op}_{\mathcal{C}}$ and X a finite set, and a collection of finite sets indexed by X, $\{Y_x | \, x \in X\}$. Define the surjection

$$h : Y = \bigsqcup_{x \in X} Y_x \twoheadrightarrow X \text{ defined by } h(y) = x \text{ for } y \in Y_x.$$

According to Definition 1.2.7 there is a morphism

$$\gamma_{X,h} : A(X) \odot A[h] \longrightarrow A(\bigsqcup_{x \in X} Y_x),$$

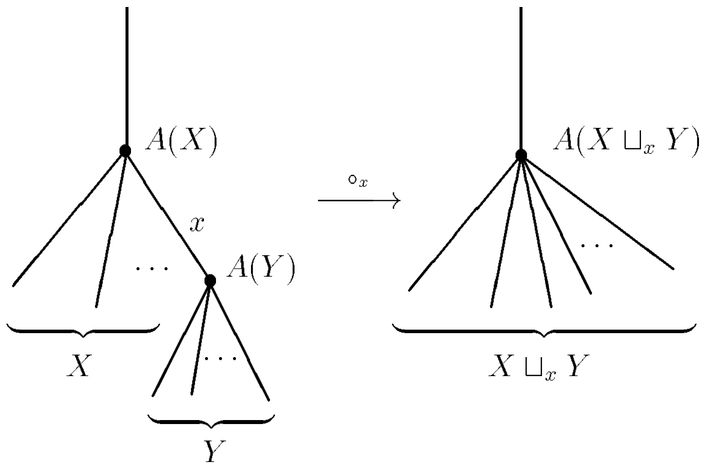


Figure 1.5: $A(\pi_x) = \circ_x$ where φ_x collapses edge x. Note also that the tree on the left is an example of a grafting along the edge x.

which, together with the isomorphism

$$\iota : A(X) \odot \bigodot_X A(Y_x) \xrightarrow{\cong} A(X) \odot A[h]$$

gives

$$\gamma_{X,Y} := \gamma_{X,h} \circ \iota : A(X) \odot \bigodot_X A(Y_x) \longrightarrow A(\bigsqcup_{x \in X} Y_x). \tag{1.30}$$

For a pseudo-operad the composition operations $\circ_x$ can be used to define the value of the functor A on elementary morphisms. If x is an edge pointing from v_1 to v_2, $X = In(v_2)$, $Y = In(v_1)$, then the follwing composite morphism is associated to a morphism of trees $\pi_x : T \to T/\{x\}$ of the type shown in Figure 1.5:

$$\begin{aligned} \bigodot_{v \in Vert(T)} A(In(v)) \cong A(X) \odot A(Y) \odot \bigodot_{\substack{v \in Vert(T) \\ v \neq v_1, v_2}} A(In(v)) \\ \xrightarrow{\circ_x \odot \mathbb{1}} A(X \sqcup_x Y) \odot \bigodot_{\substack{v \in Vert(T) \\ v \neq v_1, v_2}} A(In(v)) \longrightarrow \tilde{A}(T/\{x\}). \end{aligned} \tag{1.31}$$

where $\circ_x$ is defined in (1.22). The equivariance condition in Definition 1.20 and the universal property of the coequalizer imply the existence of a morphism:

$$\tilde{A}(\pi_x) : \tilde{A}(T) \longrightarrow \tilde{A}(T/\{x\}),$$

completing the construction of a functor $A : \mathtt{Tree} \to \mathcal{C}$ associated to a pseudo-operad A.

Theorem 1.4.4. *Given an object $A \in \mathtt{Fun}(\mathcal{S}^{op}, \mathcal{C})$, equation (1.29) defines an object $A(T)$ in $\mathcal{C}$ for every non-trivial tree $T \in \mathtt{Tree}$. Assume further that composition operations*

$$\circ_x : A(X) \odot A(Y) \longrightarrow A(X \sqcup_x Y)$$

are defined. Then if

$$\psi = \pi_{x_1} \circ \cdots \circ \pi_{x_n} : T \to T/\{x_1, x_2, \cdots x_n\}$$

is a morphism of trees factorized into a product of elementary morphisms, we define

$$A(\psi) := \circ_{x_1} \cdots \circ_{x_n} : A(T) \to A(T/\{x_1, x_2, \cdots x_n\}). \tag{1.32}$$

A necessary and sufficient condition that definition (1.32) be independent of the representation of ψ and further that $A(\psi\psi') = A(\psi)A(\psi')$ is that the composition operations $\circ_x$ satisfy the associativity axioms (1.23).

Proof. Direct inspection. ∎

Corollary 1.4.5. *The definition of A given above determines a functor if and only if the composition operations satisfy the axioms for a pseudo-operad structure on A.*

Remark 1.4.6. This theorem shows that the category of trees is really suited to the description of pseudo-operads not operads. The definition in equation (1.34) is rather gratuitous since the only morphism involving the trivial tree is the identity morphism. Furthermore, the diagrams (1.26) do not follow from the functorial properties of A, and the unit axiom in the definition of an operad is not expressed by the properties of the functor A on the category of trees.

Having described pseudo-operads in terms of trees, we introduce the operation of grafting which will allow us to construct both the free pseudo-operad and the free operad generated by an object in $\mathtt{Fun}(\mathcal{S}^{op}, \mathcal{C})$.

Definition 1.4.7. *(Grafting) Given two trees T and S and a leaf x of T, there is a tree $T \star_x S$, called the* **grafting of S to T along x**, *given by identifying the root edge of S with the leaf x of T.*

If T and S have respectively the sets of leaves X and Y, then the set of leaves for the new tree is $X \sqcup_x Y$ as defined in equation (1.21), and the set of vertices is $Vert(T \star_x S) = Vert(T) \sqcup Vert(S)$.

There is an obvious isomorphism

$$\gamma_x : A(T) \odot A(S) \xrightarrow{\sim} A(T \star_x S), \tag{1.33}$$

given by relabelling $A(T) \odot A(S)$ as an $\odot$-product over the vertices of the new tree $T \star_x S$. This operation will be the composition law for a pseudo-operad $\Psi(A)$, which we now proceed to define.

Let $\mathtt{Tree}_X$ be the set of nontrivial trees with leaves labeled by the finite set X. We have already noted in Remark 1.4.6 that the axioms of a pseudo-operad are well described by $A : \mathtt{Tree} \to \mathcal{C}$. However, the unit axiom for an operad is really an independent piece of data not expressed in terms of morphisms $A(T) \to A(T')$. We will see below that the unit axiom can be expressed using the trivial tree T_0 and defining

$$A(T_0) := \mathbb{1}. \tag{1.34}$$

for the trivial tree T_0 with no vertices: Initially we will deal with pseudo-operads; therefore, T_0 is not included in $\mathtt{Tree}_{\{x\}}$ in the definition of the free pseudo-operad. Later we consider the possibility of adjoining a unit and define the free operad.

Definition 1.4.8. *The endofunctor Ψ of $\mathtt{Fun}(\mathcal{S}^{op}, \mathcal{C})$ is defined on objects by*

$$\Psi(A)(X) := \coprod_{T \in \mathtt{Tree}_X} A(T), \text{ for a finite set } X \text{ and } A \in \mathcal{D}.$$

For $\alpha : A \to B$ and a tree T, there are morphisms $\alpha(In(v)) : A(In(v)) \to B(In(v))$ for each vertex of T, therefore, by functoriality of the the $\odot$ product there is a morphism, $\alpha(T) : A(T) \to B(T)$. Then define

$$\Psi(\alpha)(X) := \coprod_{T \in \mathtt{Tree}_X} \alpha(T) : \Psi(A)(X) \to \Psi(B)(X). \tag{1.35}$$

Theorem 1.4.9. *(Free pseudo operad) If*

$$\circ_x : \Psi(A)(X) \odot \Psi(A)(Y) \longrightarrow \Psi(A)(X \sqcup_x Y)$$

is defined on the component $A(T) \odot A(S)$ of $\Psi(A)(X) \odot \Psi(A)(Y)$ by the natural identification of $A(T) \odot A(S)$ with $A(T *_x S)$, then the object $\Psi(A)$ with compositions $\circ_x$ is a pseudo-operad.

The functor Ψ is left adjoint to the forgetful functor $\mathcal{F}_{or} : \Psi\mathtt{Op} \to \mathtt{Fun}(\mathcal{S}^{op}, \mathcal{C})$ hence defines a free pseudo-operad functor.

Proof. The pseudo-operad axioms for $\Psi(A)$ with the composition rule defined above follow from properties of the unordered $\odot$ product. By Remark 1.3.2, given that the operations $\circ_x$ are well-defined, it is enough to check the associativity conditions. The commutativity of the diagram

$$\begin{tikzcd}
A(T_1) \odot A(T_2) \odot A(T_3) \arrow[r, "\circ_{x_2}(\circ_{x_1} \odot id)"] \arrow[d, "(id \odot \tau)"'] & A((T_1 \star_{x_1} T_2) \star_{x_2} T_3) \\
A(T_1) \odot A(T_3) \odot A(T_2) \arrow[ur, "\circ_{x_1}(\circ_{x_2} \odot id)"'] &
\end{tikzcd}$$

which corresponds to the second equation in the associativity axioms (1.23) is a consequence of the definition of $A((T_1 \star_{x_1} T_2) \star_{x_2} T_3)$ as the coequalizer over the set of permutations of the vertices. The first equation in the associativity axioms (1.23) is true for the same reason.

To verify that Ψ is left adjoint to $\mathcal{F}_{or}$, we need to show that for any $A \in \mathtt{Fun}(\mathcal{S}^{op}, \mathcal{C})$ and a pseudo-operad $B \in \Psi\mathtt{Op}$ there is a bijection

$$Mor_{\Psi\mathtt{Op}}(\Psi(A), B) \underset{\xi}{\overset{\rho}{\rightleftarrows}} Mor_{\mathtt{Fun}(\mathcal{S}^{op},\mathcal{C})}(A, \mathcal{F}_{or}(B)). \tag{1.36}$$

Note that both sets of morphisms consist of natural transformations between functors from the category $\mathcal{S}$ to $\mathcal{C}$, the only difference being that in $\Psi\mathtt{Op}$ there is a condition of compatibility with the pseudo-operad composition operations which is not required in $\mathtt{Fun}(\mathcal{S}^{op}, \mathcal{C})$. According to [9, Theorem II.7.2], the adjoint relation (adjunction) $\Psi \dashv \mathcal{F}_{or}$ is equivalent to the existence of two natural transformations:

$$\zeta : id_{\mathtt{Fun}(\mathcal{S}^{op},\mathcal{C})} \mapsto \mathcal{F}_{or}\Psi \text{ and } \chi : \Psi\mathcal{F}_{or} \mapsto id_{\Psi\mathtt{Op}}, \tag{1.37}$$

called, respectively, the unit and counit of the adjunction, satisfying

$$\chi\Psi \circ \Psi\zeta = id_{\Psi} \text{ and } \mathcal{F}_{or}\chi \circ \zeta\mathcal{F}_{or} = id_{\mathcal{F}_{or}}, \tag{1.38}$$

where $\chi\Psi(A) := \chi_{\Psi(A)}$, $\Psi\zeta(A) := \Psi(\zeta_A)$, and id_Ψ is the identity natural transformation of Ψ to itself. The terms in the second equation are defined similarly.

Given ζ and χ, the bijective correspondence (1.36) is defined by

$$\rho(\alpha) := \mathcal{F}_{or}(\alpha) \circ \zeta_A, \text{ and } \xi(\beta) := \chi_B \circ \Psi(\beta)$$

for $\alpha \in Mor_{\Psi\mathtt{Op}}(\Psi(A), B)$ and $\beta \in Mor_{\mathtt{Fun}(\mathcal{S}^{op},\mathcal{C})}(A, \mathcal{F}_{or}(B))$.

After describing ζ and χ, rather than proving (1.38), we establish the bijection (1.36) directly by showing that ρ and ξ are inverse to each other.

To define $\zeta_A \in Mor_{\mathtt{Fun}(\mathcal{S}^{op},\mathcal{C})}(A, \mathcal{F}_{or}\Psi(A))$ we need a morphism

$$\zeta_A(X) : A(X) \to (\mathcal{F}_{or}\Psi(A))(X)$$

for each $X \in \mathcal{S}$. Let $t(X)$ be a corolla (the tree with one vertex, with X as the set of leaves) then define $\zeta_A(X)$ to be the imbedding of $A(X) \cong A(t(X))$ in $\mathcal{F}_{or}\Psi(A)(X)$:

$$A(X) \cong A(t(X)) \xrightarrow{\zeta(X)} \coprod_{T \in \mathtt{Tree}_X} A(T) = \Psi(A)(X).$$

Thus $\rho(\alpha)(X) : A(X) \to B(X)$ is the restriction of $\alpha(X) : \Psi(A)(X) \to B(X)$ to the component $A(t(X))$.

To define χ, note that $t(X)$ is the terminal object of $\mathtt{Tree}_X$, i.e., there is unique tree morphism $\pi : T \to t(X)$ for any tree $T \in \mathtt{Tree}_X$, which is given by contracting all internal edges of T. Then for any pseudo-operad B there is a morphism $B(\pi) : B(T) \to B(X)$. This extends to the coproduct over $T \in \mathtt{Tree}_X$ and determines a collection of $\mathcal{C}$-morphisms, one for each finite set $X \in \mathcal{S}$,

$$\chi_B(X) : \Psi\mathcal{F}_{or}(B)(X) \to B(X). \tag{1.39}$$

Together these form a morphism $\chi_B \in Mor_{\Psi\mathtt{Op}}(\Psi\mathcal{F}_{or}(B), B)$ for each $B \in \Psi\mathtt{Op}$ and define the natural transformation χ. Thus $\xi(\beta)(X) : \Psi(A)(X) \to B(X)$ is simply the extension $\Psi(\beta) : \Psi(A)(X) \to \Psi(B)(X)$ composed with the morphism $\Psi(B)(X) \to B(X)$ given by the pseudo-operad structure on B.

The equation $\rho(\xi(\beta)) = \beta$ is immediate, since ρ restricts $\xi(\beta) : \Psi(A) \to B$ to the component $A(t(X))$ of $\Psi(A)$ and the morphism $\overline{\xi(\beta)} : A(t(X)) \to B(X)$ given by the restriction of $\xi(\beta)$ to the component $A(X) \cong A(t(X))$ is just β since $t(X)$ has no edges to collapse.

To prove $\xi(\rho(\alpha)) = \alpha$ we use the compatibility condition on $\alpha \in Mor_{\Psi\mathtt{Op}}(\Psi(A), B)$ which says that the following diagram commutes

$$\begin{tikzcd}
\Psi(A)(X) \odot \Psi(A)(Y) \arrow[r, "\alpha(X) \odot \alpha(Y)"] \arrow[d, "\circ_x"] & B(X) \odot B(Y) \arrow[d, "\circ_x"'] \\
\Psi(A)(X \sqcup_x Y) \arrow[r, "\alpha(X \sqcup_x Y)"] & B(X \sqcup_x Y)
\end{tikzcd}$$

The vertical arrow on the left is the coproduct of the identifications $A(T) \odot A(S) \stackrel{\sim}{\to} A(T \star_x S)$ over trees $T \in \mathtt{Tree}_X$, $S \in \mathtt{Tree}_Y$, and any tree can be represented as a sequence of $\star_x$ products as x varies over the internal edges of the tree. Since $\Psi(A)(X)$ is a coproduct of $A(T)$ over the trees in $\mathtt{Tree}_X$, the morphism $\alpha(X)$ is determined by its restriction to the $A(T)$. For each tree T, repeated use of the diagram shows that $\alpha(X)$ is the $\odot$ product of α's acting on corollae, but this is the ξ extension of $\rho(\alpha)$, proving that $\xi(\rho(\alpha)) = \alpha$. ■

Now we consider operads. The existence of a unit morphism $\eta : \mathbb{1} \to B$ allows us to adjoin a unit to the free pseudo-operad and thus to define a free operad. In fact, the trivial tree T_0 is like an identity relative to grafting,

$$T_0 \star_x S = S = S \star_x T_0,$$

and the operation γ_x applied to $A(T_0)$ on the right or left has the property of a unit.

$$\begin{tikzcd}
A(S) \odot A(T_0) \arrow[r, "\gamma_x"] \arrow[d, "="'] & A(S \star_x T_0) \arrow[d, "="] \\
A(S) \odot \mathbb{1} \arrow[r, "\cong"] & A(S)
\end{tikzcd}
\qquad
\begin{tikzcd}
A(T_0) \odot A(S) \arrow[r, "\gamma_x"] \arrow[d, "="'] & A(T_0 \star_x S) \arrow[d, "="] \\
\mathbb{1} \odot A(S) \arrow[r, "\cong"] & A(S)
\end{tikzcd}$$

This leads us to define a new functor

$$\Gamma(A) := A(T_0) \coprod \Psi(A) = \mathbb{1} \coprod \Psi(A). \tag{1.40}$$

Proposition 1.4.10. *The functor $\Gamma(A)$ defines the free operad generated by an object $A \in$ $\mathtt{Fun}(\mathcal{S}^{op}, \mathcal{C})$.*

Proof. To show that $\Gamma(A)$ is an operad, we apply Theorem 1.3.3. If we define $\eta_x : \mathbb{1} \to \Gamma(A)(\{x\})$ by the obvious inclusion, then we have the required commutative diagrams:

$$\begin{tikzcd}
\Gamma(A)(X) \odot \mathbb{1} \arrow[r, "r"] \arrow[d, "id \odot \eta_x"'] & \Gamma(A)(X) \\
\Gamma(A)(X) \odot \Gamma(A)(\{x\}) \arrow[ur, "\circ_x"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{1} \odot \Gamma(A)(X) \arrow[r, "l"] \arrow[d, "\eta_x \odot id"'] & \Gamma(A)(X) \\
\Gamma(A)(\{x\}) \odot \Gamma(A)(X) \arrow[ur, "\circ_x"'] &
\end{tikzcd} \tag{1.41}$$

The adjointness relation

$$Mor_{\mathtt{Fun}(\mathcal{S}^{op}, \mathcal{C})}(A, \mathcal{F}_{or}(B)) \leftrightarrow Mor_{\mathtt{Op}}(\Gamma(A), B) \tag{1.42}$$

follows from Theorem 1.4.9. ∎

1.5. Collections, K-Collections, and K-operads.

In their paper on Koszul operads [7], Ginzburg and Kapranov construct the free operad generated by an $\widehat{\mathcal{S}}$-object in the category of K-collections (as defined below) for K a fixed $\mathbf{k}$-algebra. By considering K-collections, they are able to construct the free operad using a modified form of $\Psi(A)$ with tensor products over K.

The idea behind the definition of a K-collection is expressed in the next proposition, which is a useful though obvious observation.

Proposition 1.5.1. *If $A \in \mathtt{Fun}(\widehat{\mathcal{S}}^{op}, \mathcal{C})$ has a "partial operad structure" involving only the structure morphisms $\gamma_{1,1}$, $\gamma_{1,m}$, and $\gamma_{n,1,\ldots,1}$, which are assumed to satisfy the relevant operad axioms, then setting $K := A(1)$,*

$\gamma_{1,1} : K \odot K \to K$	*defines a monoid structure on K,*
$\gamma_{1,n} : K \odot A(n) \to A(n)$	*defines a left K-module structure on $A(n)$, and*
$\gamma_{n,1,\ldots,1} : A(n) \odot K^{\odot n} \to A(n)$	*defines a right $K^{\odot n}$-module structure on $A(n)$.*

In addition, the morphism $\gamma_{n,1,\dots,1}$ is equivariant with respect to the right Σ_n-action on $A(n)$ and the left Σ_n-action on $K^{\odot n}$ given by

$$\sigma \cdot (x_1 \odot \cdots \odot x_n) := x_{\sigma^{-1}(1)} \odot \cdots \odot x_{\sigma^{-1}(n)}, \text{ for } x_1, \dots, x_n \in K \text{ and } \sigma \in \Sigma_n.$$

The last sentence of the proposition follows from equivariance of the structure morphisms and is equivalent to the existence of a right action of $A[n,n]$ (see Definition 1.2.2) on $A(n)$ which extends the right action of $K^{\odot n}$:

$$A(n) \odot A[n,n] \to A(n).$$

As an example, if $\mathcal{C} = \mathrm{Mod}_{\mathbf{k}}$, then K is a $\mathbf{k}$-algebra, and $A[n,n]$ is the "twisted group algebra", $K_\tau[\Sigma_n]$, defined as the $\mathbf{k}$-module spanned by the expressions $(y_1 \otimes \cdots \otimes y_n)\sigma$ for $y_i \in K$, $\otimes = \otimes_{\mathbf{k}}$, $\sigma \in \Sigma_n$, and with multiplication given by:

$$(x_1 \otimes \cdots \otimes x_n)\,\sigma \cdot (y_1 \otimes \cdots \otimes y_n)\,\tau = \left(x_1 y_{\sigma^{-1}(1)} \otimes \cdots \otimes x_n y_{\sigma^{-1}(n)}\right)\sigma\tau.$$

Definition 1.5.2. *A collection in $\mathcal{C}$ is an object in $\mathtt{Fun}(\widehat{\mathcal{S}}^{op}, \mathcal{C})$, such that $A(1)$ is a $\mathcal{C}$-monoid and each $A(n)$ is a left $A(1)$- and right $A[n,n]$-module. For a fixed $\mathcal{C}$-monoid K, a K-collection in $\mathcal{C}$ is a collection such that $A(1) = K$. The category of collections will be denoted by $\mathtt{Col}$ and the category of K-collections by $\mathtt{Col}_K$.*

A morphism $\alpha \in Mor_{\mathtt{Col}}(A,B)$ consists of a morphism of $\mathcal{C}$ monoids $\alpha(1) : A(1) \to B(1)$ and morphisms $\alpha(n) : A(n) \to B(n)$ compatible (in the obvious sense) with the $A(1)$-$A[n,n]$ bimodule structure on $A(n)$ and the $B(1)$-$B[n,n]$ bimodule structure on $B(n)$. In the subcategory $\mathtt{Col}_K$ all morphisms satisfy, by definition, $\alpha(1) = id_K$ so we can start at degree 2 and require that $\alpha(n)$ be a morphism of K-$K[n,n]$ bimodules.

In [7] the definition of a K-collection in the category $\mathrm{Mod}_{\mathbf{k}}$, is a sequence $A = \{A(n) | \, 2 \le n < \infty\}$ such that $A(n)$ is a left K and right $K_\tau[\Sigma_n]$-module, see [7, paragraph 1.2.11]. Thus it is a special case of our definitions.

We will define a new monoidal structure for the category $\mathtt{Col}_K$. First define $\mathbb{1}_K \in \mathtt{Col}_K$ by

$$\mathbb{1}_K(n) = \begin{cases} K, & \text{for } n = 1, \text{ and} \\ \mathbf{0}, & \text{for } n \ge 2, \end{cases} \tag{1.43}$$

and let $K[n,n]$ denote $\mathbb{1}_K[n,n]$. In the new monoidal structure $\square_K$ on $\mathtt{Col}_K$ we use the multiplication $\odot_{K[n,n]}$, defined by letting $X \odot_{K[n,n]} Y$ be the coequalizer of the two morphisms

$$X \odot K[n,n] \odot Y \underset{id \odot \lambda}{\overset{\rho \odot id}{\rightrightarrows}} X \odot Y,$$

where ρ is the right $K[n,n]$-module structure on X and λ the left $K[n,n]$-module structure on Y.

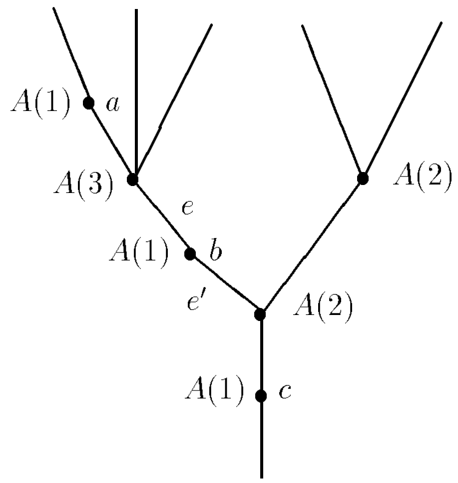


Figure 1.6: An unreduced tree with vertices of arity 1.

Suppose that A and B are K-collections; for $n \geq 2$ define

$$(A\Box_K B)(n) := \coprod_{j=1}^{n} A(j) \odot_{K[j,j]} B[j,n]. \tag{1.44}$$

Note that $B[j,n]$ is defined as before using $\odot$ so that it is a left $K[j,j]$, right $K[n,n]$-module. The unit object is defined in (1.43). The associativity of $\Box_K$ is proved just as for $\Box$.

Define a K-operad to be an operad $A \in \mathtt{Op}$ with $A(1) = K$, and denote the category of K-operads by $\mathtt{Op}_K$. Then the following proposition is self-evident.

Proposition 1.5.3. *A K-operad is a monoid in the category of $\mathtt{Col}_K$ relative to the $\Box_K$-product.*

Remark 1.5.4. The definition of a K-operad implies that the underlying $\widehat{\mathcal{S}}$-object is a K-collection. Since the identity axiom is part of the K-$K[n,n]$ bimodule structure on a K-collection, it is not necessary to adjoin $A(T_0)$ to $\Psi(A)$ in order to have an operad. Therefore Ginzburg and Kapranov can use a modification of $\Psi(A)$ to define the free operad generated by a K-collection (the left adjoint to the forgetful functor from K-operads to K-collections). Their construction involves tensor products reduced using the K-$K[n,n]$ bimodule structure. This reduction allows for a further simplification and one has only to consider reduced trees, with no vertices having one incoming edge.

1.6. GK-construction.

We now describe the Ginzburg and Kapranov (which we abbreviate GK) construction. Just as for an operad, where there is a morphism $A(T) \to A(T/e)$ contracting any internal edge e

so for a K-collection, there is a morphism contracting any internal edge which has a vertex of arity 1. A vertex of arity 1 may be adjacent to a leaf or the root edge, as in the case of the vertices a and c in Figure 1.6. In either of these cases it is adjacent to only one internal edge and there is only one choice of an internal edge to collapse. But if the vertex of arity 1 is adjacent to two internal edges as in the case of the vertex marked b in Figure 1.6, then there are two possible edges, e, e', to collapse. Both possibilities leave us with the same tree, $T/e = T/e'$. Thus we have two morphisms,

$$\begin{tikzcd} A(T) \arrow[r, bend left, "\pi_e"] \arrow[r, bend right, "\pi_{e'}"'] & A(T/e). \end{tikzcd}$$

Steve: Jim asks why reduce this way. Answer to describe the tensor product over K.

The GK reduction proceedure involves taking the coequalizer of all these morphisms. More precisely, for any reduced tree T, we can construct an unreduced tree $\hat{T}$, having one new vertex of arity 1 on each of the internal edges. Let n be the number of internal edges and $\kappa(T)$ the set of 2^n tree morphisms $\hat{T} \to T$ corresponding to all the possibilities of choosing a collapsing of one of the adjacent edges at each of the new vertices of arity 1. Define

$$A_K(T) := \text{coequalizer } \{A(\pi) : A(\hat{T}) \to A(T) | \ \pi \in \kappa(T)\}. \tag{1.45}$$

Definition 1.6.1. *(Ginzburg-Kapranov construction of a free operad) Let* $\mathtt{Tree}_X^{red}$ *be the subset of* $\mathtt{Tree}_X$ *consisting of reduced trees with leaves labeled by* X*. For* $A \in \mathtt{Col}_K$ *define*

$$\Gamma_K(A)(X) := \coprod_{T \in \mathtt{Tree}_X^{red}} A_K(T).$$

The operad structure is given by grafting.

Since the only reduced tree with one leaf is the trivial tree, it follows from equation (1.34) that

$$\Gamma_K(A)(\{x\}) := K.$$

For any tree T there is an associated reduced form T^{red} given by deleting all vertices of arity 1. By choosing a sequence of edge collapsings, one for each of the deleted vertices, we can define a morphism $\psi : A(T) \to A(T^{red})$. Of course, ψ depends on the choices we have made, but composing with the coequalizer epimorphism $\pi_{T^{red}} : A(T^{red}) \to A_K(T^{red})$, we get a morphism

$$\pi_{T^{red}} \circ \psi : A(T) \to A_K(T^{red}) \tag{1.46}$$

which is independent of the choices in the definition of ψ. Passing to the coproduct over $\mathtt{Tree}_X$ gives a morphism

$$\pi_K : \Psi(A) \longrightarrow \Gamma_K(A). \tag{1.47}$$

Theorem 1.6.2. *The functor* $\Gamma_K : \mathtt{Col}_K \longrightarrow \mathtt{Op}_K$ *is left adjoint to the forgetful functor* $\mathcal{F}_{or} : \mathtt{Op}_K \longrightarrow \mathtt{Col}_K$*, hence defines the free operad functor on* $\mathtt{Col}_K$*.*

Proof. One can give a direct proof similar to the proof of Theorem 1.4.9 or one can prove this as a corollary of that theorem. If $A \in \mathtt{Col}_K$ then $\Psi(A)$ is nearly a K-operad, except that

$$\Psi(A)(1) = K \sqcup K^{\odot 2} \sqcup K^{\odot 3} \cdots .$$

Modulo this fact, π_K defined in equation (1.47) is a K-operad morphism. Given $A \in \mathtt{Col}_K$, $B \in \mathtt{Op}_K$ we have the bijection in (1.36) and an inclusion in the right hand term of (1.36):

$$j : Mor_{\mathtt{Col}_K}(A, \mathcal{F}_{or}(B)) \hookrightarrow Mor_{\mathtt{Fun}(\mathcal{S}^{op},\mathcal{C})}(A, \mathcal{F}_{or}(B)),$$

and an injection into the left hand term of (1.36):

$$\pi_K^* : Mor_{\mathtt{Op}_K}(\Gamma_K(A), B) \to Mor_{\Psi\mathtt{Op}}(\Psi(A), B)$$

given by composition with π_K. The existence of a bijection between $Mor_{\mathtt{Col}_K}(A, \mathcal{F}_{or}(B))$ and $Mor_{\Psi\mathtt{Op}_K}(\Gamma_K(A), B)$ follows once we show that the image of π_K^* corresponds to the image of j under the bijection in (1.36). The image of π_K^* consists of the pseudo-operad morphisms β which factor as $\beta = \alpha \circ \pi_K$. In the particular case of a finite set $[n]$, let $t(n)$ be the corresponding corolla, T the unreduced tree constructed from $t(n)$ by adding a vertex of arity 1 on each edge, and $\psi : T \to t(n)$ the obvious morphism of trees. The factorization of β gives a commutative diagram

$$\begin{tikzcd} A(T) \arrow[r, "\overline{\beta(T)}"] \arrow[d, "\pi_K = A(\psi)"'] & B(T) \arrow[d, "B(\psi)"] \\ A(n) = A(t(n)) \arrow[r, "\overline{\alpha}"] & B(t(n)) = B(n) \end{tikzcd}$$

where $\overline{}$ indicates the restriction to the appropriate component. This is equivalent to requiring that $\rho(\alpha)(n) \in Mor_{\mathcal{C}}(A(n), \mathcal{F}_{or}(B)(n))$ intertwine the K-$K[n,n]$ bimodule structures on $A(n)$ and $\mathcal{F}_{or}(B)(n)$, but that exactly describes the image of j. Thus the bijection in (1.36) restricts to

$$Mor_{\mathtt{Op}_K}(\Gamma_K(A), B) \leftrightarrow Mor_{\mathtt{Col}_K}(A, \mathcal{F}_{or}(B)). \qquad \blacksquare$$

$$
\begin{tikzcd}
TTT \arrow[r, "T\mu"] \arrow[d, "\mu T"'] & TT \arrow[d, "\mu"] \\
TT \arrow[r, "\mu"] & T
\end{tikzcd}
\qquad
\begin{tikzcd}
T \arrow[rr, "T\upsilon"] \arrow[dr, "id"'] & & TT \arrow[dl, "\mu"] \\
& T &
\end{tikzcd}
\qquad
\begin{tikzcd}
T \arrow[rr, "\upsilon T"] \arrow[dr, "id"'] & & TT \arrow[dl, "\mu"] \\
& T &
\end{tikzcd}
$$

Figure 1.7: Associativity and unit axioms for a triple.

1.7. Triples.

In this section we review the definition of a triple (monad) and give a description of pseudo-operads and operads in terms of triples and algebras over a triple. The relevant triples come from the endofunctors Ψ, Γ, and Γ_K.

Let $End(\mathcal{C})$ be the strict symmetric monoidal category of endofunctors on $\mathcal{C}$ where multiplication is composition of functors, $\mathbb{1}$ is the identity functor, and all of r, l, a in are identity morphisms.

Definition 1.7.1. *A triple (also called a monad) T on a category $\mathcal{C}$ is a associative and unital monoid (T, μ, υ) in $End(\mathcal{C})$. The multiplication μ and unit morphism υ satisfy the axioms given by commutativity of the diagrams in Figure 1.7.*

Triples arise naturally from a pair of adjoint functors. Given an adjoint pair,

$$
\begin{tikzcd}
\mathcal{A} \arrow[r, bend left, "G"] & \mathcal{B} \arrow[l, bend left, "F"]
\end{tikzcd}, \text{ with associated bijection } Mor_{\mathcal{A}}(F(X), Y) \leftrightarrow Mor_{\mathcal{B}}(X, G(Y)),
$$

there is a triple in $\mathcal{B}$ defined by $T = GF$. The natural transformation $id \to GF$ (unit of the adjunction) defines the unit υ of the triple and the natural transformation $FG \to id$ (counit of the adjunction) induces a natural transformation $GFGF \to GF$ which defines the multiplication μ. In fact, it is a theorem of Eilenberg and Moore, see [25], that all triples arise this way from adjoint pairs.

Definition 1.7.2. *A T-algebra (A, α) or algebra over the triple T is an object A of $\mathcal{C}$ together with a structure morphism $\alpha : T(A) \to A$ satisfying $\alpha(T(\alpha)) = \alpha(\mu_A)$ and $\alpha\upsilon_A = id_A$. See Figure 1.8.*

Morphisms of T-algebras must commute with the structure maps. The category of T-algebras in $\mathcal{C}$ will be denoted $\mathtt{Alg}_T(\mathcal{C})$.

$$
\begin{tikzcd}
T(T(A)) \arrow[r, "T(\alpha)"] \arrow[d, "\mu"'] & T(A) \arrow[d, "\alpha"] \\
T(A) \arrow[r, "\alpha"] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[rr, "\upsilon_A"] \arrow[dr, "id"'] & & T(A) \arrow[dl, "\alpha"] \\
& A &
\end{tikzcd}
$$

Figure 1.8: T-algebra structure

Since the functors Ψ and Γ are left adjoints to $\mathcal{F}_{or} : \Psi\mathtt{Op} \to \mathtt{Fun}(\mathcal{S}^{op}, \mathcal{C})$ and $\mathcal{F}_{or} : \mathtt{Op} \to \mathtt{Fun}(\mathcal{S}^{op}, \mathcal{C})$, respectively, the functors $\mathcal{F}_{or}\Psi$ (denoted simply Ψ) and $\mathcal{F}_{or}\Gamma$ (denoted Γ) define triples on $\mathtt{Fun}(\mathcal{S}^{op}, \mathcal{C})$. Similarly, Γ_K is left adjoint to $\mathcal{F}_{or} : \mathtt{Op}_K \to \mathtt{Col}_K$, and there is a corresponding triple $\mathcal{F}_{or}\Gamma_K$ (denoted Γ_K) on $\mathtt{Col}_K$.

In the case of the triple Γ, the multiplication μ_Γ is the natural extension of μ_Ψ to $\Gamma(A) = \mathbb{1} \coprod \Psi(A)$. We can use the right and left unit morphisms $B \odot \mathbb{1} \to B$ and $\mathbb{1} \odot B \to A$, respectively, to define

$$\iota : \Psi(\mathbb{1} \coprod \Psi(A)) \to \Psi(\Psi(A)), \tag{1.48}$$

and use ι to define μ_Γ.

Definition 1.7.3. *The multiplication μ_Ψ is defined by setting $B = \Psi(A)$ in equation (5.6.1),*

$$\mu_\Psi : \Psi(\Psi(A)) \xrightarrow{\chi_{\Psi(A)}} \Psi(A). \tag{1.49}$$

The multiplication μ_Γ for the triple Γ is the composite

$$\Gamma(\Gamma(A)) = \mathbb{1} \coprod \Psi\left(\mathbb{1} \coprod \Psi(A)\right) \xrightarrow{(id,\iota)} \mathbb{1} \coprod \Psi(\Psi(A)) \xrightarrow{(id,\mu_\Psi)} \mathbb{1} \coprod \Psi(A) = \Gamma(A). \tag{1.50}$$

For the triple Γ_K, the multiplication μ_{Γ_K} is the natural reduction of μ_Ψ to $\Gamma_K(\Gamma_K(A))$ using the structure of a K-operad on $\Gamma_K(A)$.

We describe μ_Ψ in a little more detail. If we write

$$
\begin{aligned}
\Psi(\Psi(A))(X) &= \coprod_{T \in \mathtt{Tree}_X} \Psi(A)(T) = \coprod_{T \in \mathtt{Tree}_X} \bigodot_{v \in Vert(T)} \Psi(A)(In(v)) \\
&= \coprod_{T \in \mathtt{Tree}_X} \bigodot_{v \in Vert(T)} \left(\coprod_{S_v \in \mathtt{Tree}_{In(v)}} A(S_v) \right),
\end{aligned}
$$

and then move the $\odot$ product past the coproduct, the result is the coproduct

$$\coprod_{\{(T,S_v) | T \in \mathtt{Tree}_X,\, S_v \in \mathtt{Tree}_{In(v)}\}} \left(\bigodot_{v \in Vert(T)} A(S_v) \right).$$

Let $T \star \{S_v | v \in Vert(T)\}$ be the new tree given by replacing the vertex v of T with a tree S_v having $In(v)$ leaves. Then the coproduct component

$$\bigodot_{v \in Vert(T), S_v \in \mathtt{Tree}_{In(v)}} A(S_v) \cong \bigodot_{w \in V(T \star \{S_v | v \in Vert(T)\})} A(In(w)) = A(T \star \{S_v | v \in Vert(T)\}),$$

corresponding to the index (T, S_v) is also a component of $\Psi(A)(X)$, and we get for each finite set X and for each component of $\Psi(\Psi(A))(X)$ a morphism into $\Psi(A)(X)$. This morphism generates μ_Ψ.

Proposition 1.7.4. *An object $A \in \mathtt{Fun}(\mathcal{S}^{op}, \mathcal{C})$ is a Ψ-algebra if and only if it is a pseudo-operad. An object $A \in \mathtt{Fun}(\mathcal{S}^{op}, \mathcal{C})$ is a Γ-algebra if and only if it is an operad. An object $A \in \mathtt{Col}_K$ is a Γ_K-algebra if and only if it is a K-operad.*

Proof. We outline the proof of the implication in one direction, from algebra to pseudo-operad or operad. The converse is left as an exercise for the reader. Let A be a Ψ-algebra. Restriction of the structure morphism $\alpha : \Psi(A) \longrightarrow A$ to the components of $\Psi(A)$ given by trees with one internal edge defines the pseudo-operad composition maps, $\circ_x$. See Figure 1.5. The associativity axioms follow from the condition $\alpha\Psi(\alpha) = \alpha\mu_\Psi$. The pictorial proof of a particular case of associativity is given in Figure 1.9. ¿From this example the general argument is clear, but writing out the details would be tedious.

The structure morphism $\alpha : \Gamma(A)(\{x\}) = \mathbb{1} \coprod A(\{x\}) \to A(\{x\})$ restricted to the component $\mathbb{1}$ is the unit morphism $\eta_x : \mathbb{1} \to A(\{x\})$. We will prove the unit axiom for an operad as expressed in the commutative diagrams (1.26). Consider a tree T with two vertices, one of arity 1 and one of arity n, and the component of $\Gamma(A)(T)$ of $\Gamma(\Gamma(A))$. Label the vertex of arity 1 by $\mathbb{1}$ and the vertex of arity n by $A(t(X))$ (a component of $\Gamma(A)(X)$) where X is the set of incoming edges. Consider for the moment the ordered $\odot$ product with the label on the root vertex appearing first. In this case the condition $\alpha\Gamma(\alpha) = \alpha\mu_\Gamma$ is expressed by the diagrams below. The arrow $\Gamma(\alpha)$ applies the unit morphism $\eta_x : \mathbb{1} \to A(\{x\})$ at the vertex of arity 1 and the identification $A(t(X)) = A(X)$ at the other vertex. According to Definition 1.50, the restriction of μ_Γ to the component $\mathbb{1} \odot A(t(X))$ or $A(t(X)) \odot \mathbb{1}$ applies the unit axiom for the $\odot$ product to remove the factor $\mathbb{1}$. The vertical arrows on the right are both the identification $A(t(X)) = A(X)$ and the bottom arrow is $\circ_x$. Thus these two diagrams are equivalent to the diagrams (1.26).

$$\begin{tikzcd}
A(t(X)) \odot \mathbb{1} \arrow[r, "\mu_\Gamma"] \arrow[d, "\Gamma(\alpha)"'] & A(t(X)) \arrow[d, "\alpha"] & \mathbb{1} \odot A(t(X)) \arrow[r, "\mu_\Gamma"] \arrow[d, "\Gamma(\alpha)"'] & A(t(X)) \arrow[d, "\alpha"] \\
A(X) \odot A(\{x\}) \arrow[r, "\alpha"] & A(X) & A(\{x\}) \odot A(X) \arrow[r, "\alpha"] & A(X)
\end{tikzcd}$$

The implication from Γ_K-algebra to K-operad is essentially the same as the implication from Γ-algebra to operad. ■

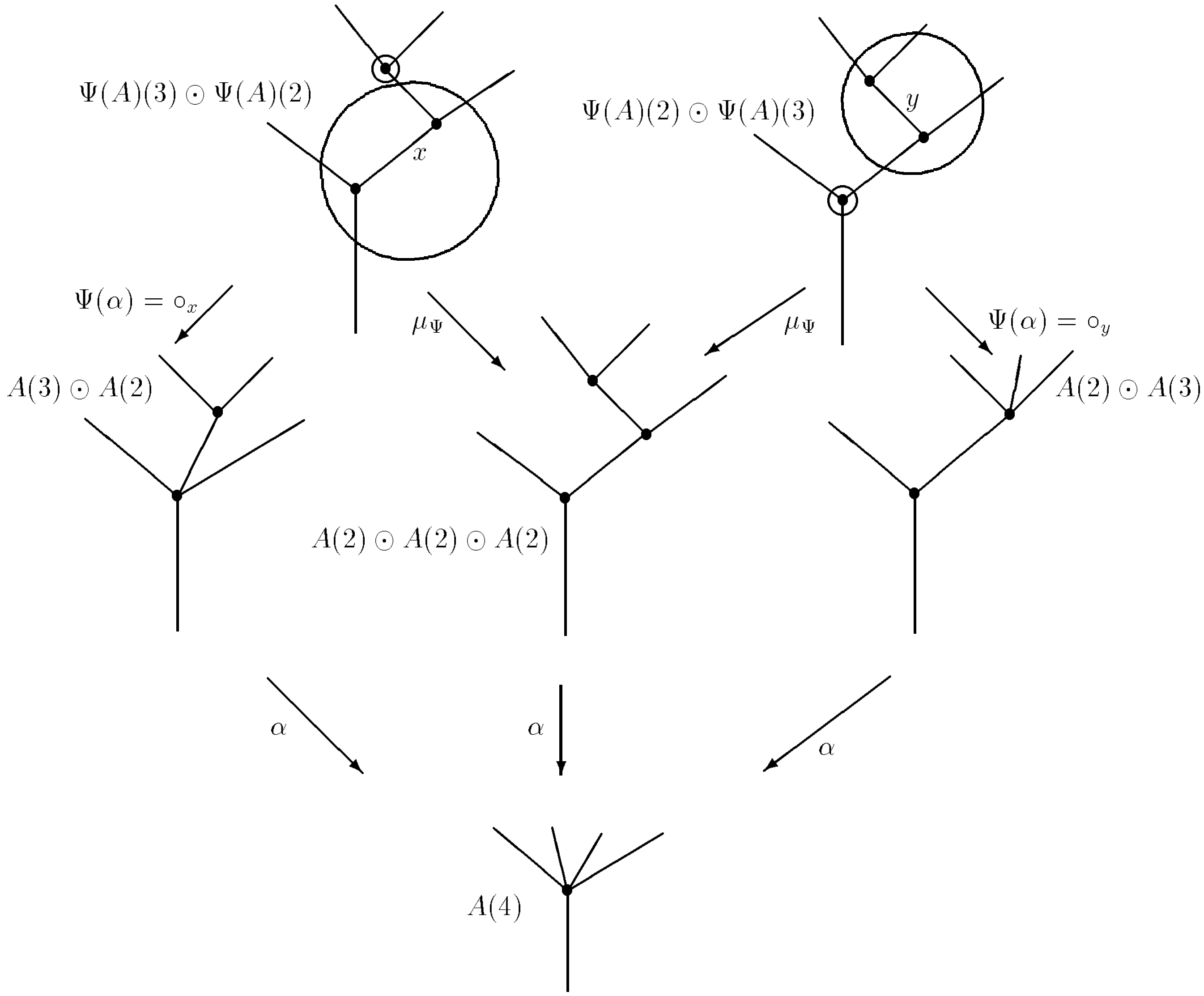


Figure 1.9: An example showing how the associativity axiom $\circ_x(id \odot \circ_y) = \circ_y(\circ_x \odot id)$ in the case $A(2) \odot A(2) \odot A(2)$ follows from $\alpha\mu_\Psi = \alpha\Psi(\alpha)$ applied twice to two different elements of $\Psi(\Psi(A))$ with the same image under μ_Ψ.

Chapter 2

Topology – Review of Classical Results

2.1. Iterated loop spaces.

Version of May 8, 2000

May introduced the concept and terminology of operads to study iterated loop spaces. Here we provide an overview of the main techniques of several authors, referring to the original papers for some of the more technical details.

By an iterated loop space, we mean a k-fold loop space for $1 < k < \infty$ or an infinite loop space. Let us remark that the standard terminology is an n-fold loop space, but since in our book the letter 'n' almost exclusively denotes the arity, we decided, to avoid confusion, to change the terminology a bit.

Definition 2.1.1. *An* k-fold loop space $\Omega^k X$ *is the space of based maps of the sphere* S^k *to a space* X*,* $1 \leq k < \infty$*. Equivalently,* $\Omega^k X = \{\lambda : (I^k, \partial I^k) \to (X, *)\}$*.*

It is helpful to interpret 'k-fold loop space' as the sequence $\{Y_i = \Omega Y_{i+1} |\ 1 \leq i < k\}$ *with*

$$Y_k = X, Y_{k-1} = \Omega X, \ldots, Y_1 = \Omega^k X.$$

An infinite loop space *$(k = \infty)$ is then a sequence* $\{Y_i = \Omega Y_{i+1} |\ 1 \leq i\}$*.*

May emphasized three uses of operads in this study: a recognition principle, a 'geometric' approximation construction and a theory of homology operations. His recognition principle carried further the work of Stasheff [?] recognizing loop spaces, Beck [?] for k-fold loop spaces and Boardman-Vogt for k-fold [?] and infinite loop spaces [?].

By a *recognition principle* (in the strong sense) for k-fold loop spaces is meant a specification of appropriate structure on a space Y such that Y possesses this structure if and only if Y has the homotopy type of a k-fold loop space. In the cases we consider, the structure is specified as that of an algebra over a certain operad. There are operads, cf. the little k-cubes operad below (Definition 2.2.1), for which the algebras are of the homotopy type of a k-fold loop space, but for which the reverse implication may fail. For that inverse implication, one must replace the operad by an equivalent cofibrant one [?]. One may use for example a functorial cofibrant replacement provided by the W-construction of Boardman and Vogt [?].

For $k = 1$, recognition is provided by the non-Σ operad $\mathcal{K}$ of Stasheff **??**, which happens to be cofibrant (Theorem **??**) and which is in fact in a sense a minimal cofibrant replacement for the little intervals operad (cf. Section **??**).

For the two extreme cases, $k = 1$ and $k = \infty$, we can characterize the appropriate operad whose algebras have the homotopy type of an n-fold loop space quite simply as a non-Σ operad with each component contractible, respectively as a Σ-operad with each component contractible and with a free Σ-action [?]. For $k = 2$, Fiedorowicz [?] has recently given a characterization of appropriate operads in terms of a braided structure. For $2 < k < \infty$, the question is open.

Geometric *approximation* constructs a space of the homotopy type of $\Omega^k S^k X$ for $1 \leq k < \infty$ or

$$\Omega^\infty S^\infty X := \varinjlim \Omega^s S^s X \quad \text{for } k = \infty,$$

built from products of X and certain standard spaces depending only on k. For $k = 1$, this was done by James [?], for $k < \infty$ by Milgram [?], Boardman and Vogt? [?], May [?], Segal [?] and for $k = \infty$ first by Dyer and Lashof [?] and later by Barratt [?], Quillen [?] and Segal [?]. The standard spaces are related to operads and include classical configuration spaces, the permutahedra and Milgram's polyhedra built from permutahedra as well as May's construction in terms of the little cubes operads.

Prior to his introduction of operads, May in [89] reformulated the algebra behind Steenrod's work so that it applied also to the homology of iterated loop spaces. There the operations were first introduced, mod 2, by Kudo and Araki [?] and, mod $p > 2$, by Dyer and Lashof [24]. The notions of operad and especially E_∞-operad and E_∞-space simplify the construction, analysis and understanding of these operations and those of Steenrod as well as there relation. May in [66, page 33] wrote: "Actually, [89] should be read as a paper about operad actions. Unfortunately, it was written shortly before operads were invented."

Although the simplification is somewhat less substantial, the use of operads is also relevant to our understanding of Massey operations **??**, originally defined in cohomology [?, ?] and their generalizations [?, ?, 104].

2.2. Recognition.

Version of May 8, 2000

For $k = 1$, recognition is provided by the non-symmetric operad $\mathcal{K}$ of Stasheff, c.f. Theorem **??**, while for $1 < k < \infty$, an algebra over the little k-cubes operad invented by Boardman and Vogt is of the homotopy type of a k-fold loop space. The little k-cubes operad is defined as follows.

Let I^k denote the standard k-dimensional unit cube. *A little k-cube* is a linear embedding $c : I^k \hookrightarrow I^k$ with parallel axes, that is, the components of

$$c(t_1, \ldots, t_n) = (c^1(t_1), c^2(t_2), \ldots, c^n(t_n)),\ t_i \in I \text{ for } 1 \leq i \leq n,$$

are linear functions $c^i : I \to I$ of the form $c^i(t) = (y_i - x_i)t + x_i$ with $0 \leq x_i < y_i \leq 1$, $1 \leq i \leq n$. This means that each function c^i consists of a shrinking by the factor $y_i - x_i$ and translation by x_i.

Definition 2.2.1. *The* little k-cubes operad $\mathcal{C}_k = \{\mathcal{C}_k(n)\}_{n\geq 0}$ *consists of the spaces* $\mathcal{C}_k(n)$ *of n-tuples* $(c_n, \ldots, c_n)$ *of little k-cubes such that the images* $c_1(\overset{\circ}{I}{}^k), \ldots, c_n(\overset{\circ}{I}{}^k)$ *of the interior* $\overset{\circ}{I}{}^k$ *of* I^k *are mutually disjoint. For* $j = 0$, $\mathcal{C}_k(0)$ *is defined to be the one point space.*

As topological space, $\mathcal{C}_k(n)$ can be identified with an open subspace of $\mathbb{R}^{2nk}$ using the $\{y_i, x_i\}_{i=1}^k$ as coordinates. The operad structure is given fairly obviously: the symmetric group Σ_n acts on $\mathcal{C}_k(n)$ by permuting the labels of little cubes and the structure map γ is given by composition of embeddings. To be more precise, let $c : \bigsqcup_{s=1}^n I_s^k \to I^k$ and $c_s : \bigsqcup_{j=1}^{m_s} I_{j,s}^k \to I^k$ (where I_s^k and $I_{j,s}^k$ are identical copies of the standard k-cube) be elements of $\mathcal{C}_k(n)$, resp. $\mathcal{C}_k(m_s)$, $1 \leq s \leq n$. Then $\gamma(c; c_1, \ldots, c_n) \in \mathcal{C}_k(m_1 + \cdots + m_n)$ is the map

$$\gamma(c; c_1, \ldots, c_n) : \bigsqcup_{s=1}^{n} \bigsqcup_{j=1}^{m_s} I_{j,s}^k \to I^k$$

is given by

$$\gamma(c; c_1, \ldots, c_n)|_{I_{j,s}^k} := c \circ c_s,$$

where we interpret c_s as a map $c_s : \bigsqcup_{j=1}^{m_s} I_{j,s}^k \to I_s^k$. The composition law is illustrated in Figure 2.1. The unit $1 \in \mathcal{C}_k(1)$ is the identity map $\mathbb{1}_{I^k} : I^k \to I^k$.

The operad $\mathcal{C}_1$ is called also the *little intervals operad*, while $\mathcal{C}_2$ is called simply the little cubes operad and denoted $\mathcal{C}$. The 'infinite' version of $\mathcal{C}_k$ is introduced as follows.

Definition 2.2.2. *The operad* $\mathcal{C}_\infty = \{\mathcal{C}_\infty(n)\}_{n\geq 0}$ *is defined by* $\mathcal{C}_\infty = \varinjlim\ \mathcal{C}_k$ *with respect to the morphisms of operads* $\sigma_k : \mathcal{C}_k \to \mathcal{C}_{k+1}$ *given by*

$$\sigma(n)(c_1, \ldots, c_n) := (c_1 \times \mathbb{1}_I, \ldots, c_n \times \mathbb{1}_I),\ n \geq 1.$$

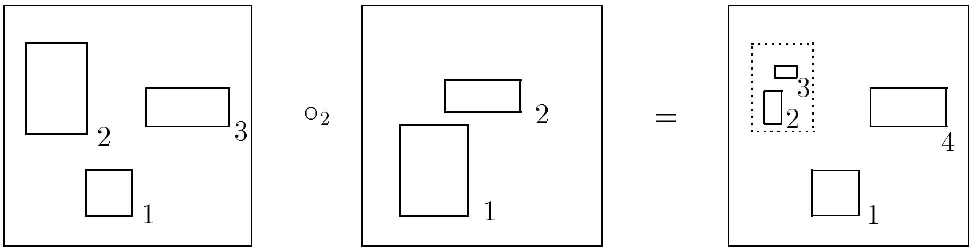


Figure 2.1: The $\circ_2$-product of an element of $\mathcal{C}_2(3)$ with an element of $\mathcal{C}_2(2)$. The result is an element of $\mathcal{C}_2(4)$.

Boardman and Vogt constructed an action $A_{BV} : \mathcal{C}_k \to End_X$ of the little k-cubes operad $\mathcal{C}_k$ on a k-fold loop space as follows. Given an n-tuple $\lambda_i : (I^k, \partial I^k) \to (X, *) \in \Omega^k X$, $1 \leq i \leq n$, and a little k-cube $c = (c_1, \ldots, c_n) \in \mathcal{C}_k(n)$, the action $A_{BV}(c)(\lambda_1, \ldots, \lambda_n)$ is the map defined to be λ_i (suitably rescaled) on the image of c_i and to be $*$ on the complement of all the images of the maps c_i.

Theorem 2.2.3. *(Boardman-Vogt [?], May [?]) A connected $\mathcal{C}_k$-space X is of the homotopy type of a k-fold loop space.*

As for Stasheff's result recognizing a space of the homotopy type of a loop space by an action of the operad $\mathcal{K}$ of associahedra, a 'de-looping' is constructed explicitly. For this purpose, we use further generalizations of bar constructions.

2.3. The bar construction: theme and variations.

Version of May 10, 2000

The bar construction occurred first in the work of Eilenberg and MacLane [?] in their extension of Hopf's work [?], one of the earliest works on the cohmology of groups. Originally invented for the group ring of a discrete group, it has been reincarnated time and again in a variety of categories: differential graded algebraic [?, ?], topological [?, ?, ?], categorical [?, ?]. As far as operads are concerned, there is a bar construction for algebras over a given operad $\mathcal{O}$ and also a bar construction for operads themselves.

To be more precise, the term 'bar construction' is ambiguous, even in a specified category. Originally there was an acyclic bar construction BA for an algebra A over a ground ring R which was free as an A-module generated by an R-module $\overline{B}A$, the reduced bar construction. Over time, with increased attention paid to the reduced bar construction (cf. a classifying space versus the corresponding universal principal bundle), the notation BA was commonly used for the reduced construction; we will follow this convention.

These two variants and more are covered by the general *'two-sided bar construction'* $B(L, A, R)$ which we now describe in a way that works equally well if

(i) A is an algebra (ordinary or dg) with right module R and left module L,
(ii) A is a monoid (abstract or topological) with right A-space R and left A-space L,
(iii) L, A, R are categories with functors $A \to R$ and $A \to L$,
(iv) A is itself an operad and R and L are either modules over A or algebras over A or one of each, or
(v) A is a monad (= triple), L is an A-functor and R is an A-space.

We assume we are working in a category with finite products. Given the data L, A, R, form the facial (pre-simplicial) object $B_\bullet(L, A, R)$ which in degree n is $L \times A^{\times n} \times R$, with face maps
$$d_i : B_n(L, A, R) \to B_{n-1}(L, A, R)$$
given by

– the structure map on $L \times A$ if $i = 0$,
– the structure map on $A \times A$ if $1 \leq i < n$, and
– the structure map on $A \times R$ if $i = n$.

Then the two-sided bar construction $B(L, A, R)$ is the realization of $B_\bullet(L, A, R)$ in the original category.

Three special cases are of particular importance, corresponding to the classical universal bundle for a topological group or monoid G and a point $*$:
$$B(G, G, G) \simeq G \to B(G, G, *) = EG \to B(*, G, *) = BG.$$
In the general setting, $*$ is the terminal object in the category, so the A-action is trivial. The contractibility of the universal total space EG is a special case of the (weak) equivalence
$$X \leftarrow B(A, A, X) \tag{2.1}$$
which holds for any A-object X.

Now we are ready to sketch a proof of Theorem 2.2.3 according to May. Boardman and Vogt in their book [?] used a somewhat different iterative method.

As we already explained **??**, each operad $\mathcal{O}$ defines a triple O : `Top` $\to$ `Top` with the property that $\mathcal{O}$-algebras are precisely algebras over the triple O. In particular, the little k-cubes operad $\mathcal{C}_k$ induces the triple C_k. Given a triple O, May introduced O-functors as functors on which the triple O acts in an appropriate sense, see [?, page 86]. The triple O is a functor over itself.

May considered two important C_k-functors, the functor $S^k : \mathtt{Top} \to \mathtt{Top}$ assigning to each topological space X its k-fold suspension S^k, and the functor $\Omega^k S^k : \mathtt{Top} \to \mathtt{Top}$ assigning to each topological space X the k-fold iterated loop space of its k-fold suspension $\Omega^k S^k X$. He then proved [?, Theorem 13.1] that there is a sequence of (weak) homotopy equivalences

$$X \leftarrow B(C_k, C_k, X) \to B(\Omega^k S^k, C_k, X) \to \Omega^k B(S^k, C_k, X). \tag{2.2}$$

The map on the left is a homotopy equivalence because of the general property (2.1). The middle map is a homotopy equivalence because the functor $\Omega^k S^k$ is, in a suitable sense, homotopy equivalent to C_k, as follows from the approximation theorem **??**. Finally, the right map is a homotopy equivalence as a consequence of a certain interchange rule between the geometric realization and the functor Ω. Sequence (2.2) then *explicitly* represents X as the k-fold loop space of $B(S^k, C_k, X)$.

2.4. Approximation.

Version of May 8, 2000

The first approximation theorem of the type mentioned Section **??** is James' reduced product construction JX, the free topological monoid generated by X, as a model for $\Omega S X$ for a well-pointed space X:

$$JX := \coprod_{n \geq 0} X^n / \sim,$$

Here $X^0 = *$ and the equivalence relation is defined by

$$X^n \ni (x_1, \dots, x_i = *, \dots, x_n) \sim (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \in X^{n-1},$$

for any $n \geq 1$ and $1 \leq i \leq n$.

We are going to describe approximations similar to that of James for iterated loop spaces. In proving his cobar construction models the chains on a loop space, Adams [?] gave a combinatorial map from the cube I^n to the space of paths in the simplex Δ^{n+1} from the first to the last vertex. Milgram generalizes this by giving a combinatorial map from the polytope known as the permutahedron P_n to the space of paths in the cube I^n from $(0, \dots, 0)$ to $(1, \dots, 1)$. The *permutahedron* P_n can be described as an $n-1$-dimensional convex polytope with $n!$ vertices, one for each permutation of n variables.

Definition 2.4.1. *As a subspace of* $\mathbb{R}^n$, *the* permutahedron P_n *is the convex hull of the permutation group* Σ_n *itself embedded into* $\mathbb{R}^n$ *by its action on the point* $(1, 2, \dots, n) \in \mathbb{R}^n$.

As a cellular complex, the permutahedron P_n is the realization of the poset $\mathfrak{P}_n$ of ordered partitions of the set $\{1,2,\dots,n\}$. To be precise, the poset $\mathfrak{P}_n$ consists of ordered partitions $(j_1,\dots,j_w)$, $j_i \subset \{1,2,\dots,n\}$, $1 \le w \le n$. The partial order is generated by

$$(j_1,\dots,j_w) < (j_1,\dots,j_i \cup j_{i+1},\dots,j_w),\ 1 \le i < w.$$

Geometrically, the cell corresponding to $(j_1,\dots,j_w) \in \mathfrak{P}_n$ (and denoted again $(j_1,\dots,j_w)$) is the convex hull of the subgroup leaving the decomposition $(j_1,\dots,j_w)$ fixed. It is immediate to see form this description that the face $(j_1,\dots,j_w)$ has dimension $n-w$. There is a unique maximal element $(j_1) \in \mathfrak{P}_n$, $j_1 = \{1,2,\dots,n\}$ and $n!$ minimal elements (vertices) indexed by permutations

$$\Sigma_n \ni \sigma \longleftrightarrow (\sigma(1),\dots,\sigma(n)) \in \mathfrak{P}_n. \tag{2.3}$$

The poset structure of $\mathfrak{P}_3$ is illustrated in Figure 2.2. The polyhedron P_4 is portraited on

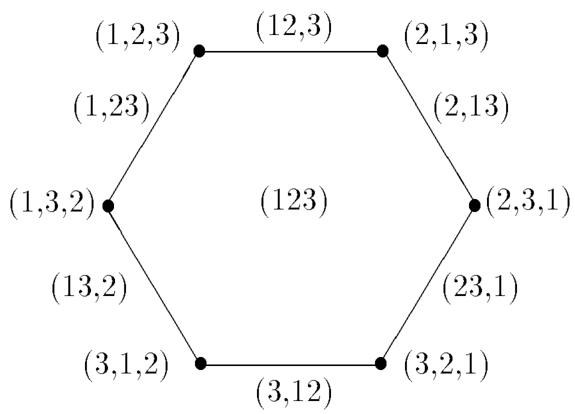


Figure 2.2: The poset structure of $\mathfrak{P}_3$.

Figure **??**.

It is clear from the definition of the permutahedron that each P_n is acted on by the symmetric group Σ_n. The natural *left* action is given on vertices by

$$\tau(\sigma(1),\dots,\sigma(n)) := (\tau\sigma(1),\dots,\tau\sigma(n)),$$

for $\sigma,\tau \in \Sigma_n$. Since we would like to consider the sequence $P = \{P(n)\}_{n\ge 1}$ as a collection, we must, to comply with the conventions accepted in this book, to flip the above action to the *right* action setting

$$(\sigma(1),\dots,\sigma(n))\tau := (\tau^{-1}\sigma(1),\dots,\tau^{-1}\sigma(n)).$$

It is clear that this action respects the cellular structure, giving rise to a cellular right Σ_n-action on P_n for each $n \ge 1$.

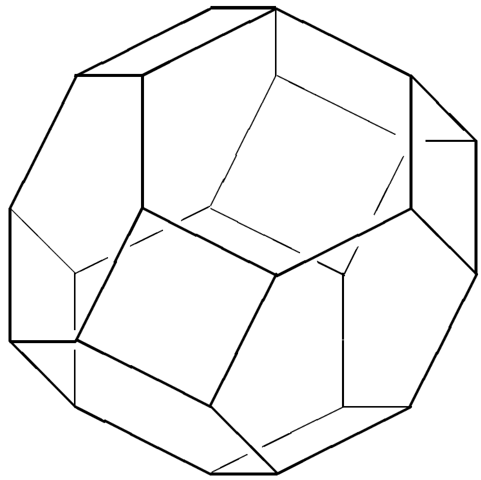


Figure 2.3: The polyhedron P_4.

Recall that the collection $\Sigma := \{\Sigma_n\}_{n\geq 0}$ has an operad structure given by its identification with the operad $\mathcal{M}on$ for topological monoids. This induces a corresponding operad structure on $\mathcal{P} = \{P_n\}$ by taking convex hulls (affine extensions in Berger's terminology), but it this structure is not cellular with respect to the above cells. For example, consider the 1-cell (12) of P_2 and describe the image $(12)\circ_2(12)$ in P_3. To this end, we must understand how $\circ_2$ acts on the vertices $(1,2)$ and $(2,1)$ of (12). It can be easily calculated that

$$(1,2)\circ_2(1,2) = (1,2,3),\quad (1,2)\circ_2(2,1) = (1,3,2),$$
$$(2,1)\circ_2(1,2) = (2,3,1),\quad (2,1)\circ_2(2,1) = (3,2,1),$$

therefore the image $(12)\circ_2(12)$ is the rectangle depicted in Figure 2.4.

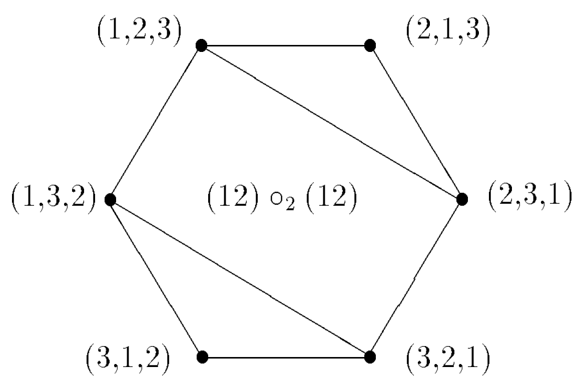


Figure 2.4: The non-cellularity of the operad structure on $P = \{P(n)\}_{n\geq 1}$.

Milgram builds [?] an approximation to $\Omega^k S^k X$ using the permutahedra, though not in an obviously operadic way. He defines, for $k \geq 1$ and $n \geq 1$, spaces

$$J_n^{(k)} =: ((P_n)^{k-1} \times \Sigma_n)/ \sim, \tag{2.4}$$

where the equivalence relation identifies certain boundary cells. For $k = 1$, $J_n^{(1)}$ reduces to a point. Let us explicitly describe the space $J_n^{(2)}$. Recall that the faces of the permutahedron P_n are indexed by decompositions $(j_1, \ldots, j_w)$ of the set $\{1, \ldots, n\}$. Denote $a_i := \mathrm{card}(j_i)$. Then the decompositions $(j_1, \ldots, j_w)$ are in one-to-one correspondence with elements $\tau \in \Sigma_n$ with the property that

$$\tau(a_1 + \cdots + a_{s-1} + 1) < \tau(a_1 + \cdots + a_{s-1} + 2) < \cdots < \tau(a_1 + \cdots + a_s), \text{ for } 1 \leq s \leq w, \tag{2.5}$$

that is, τ is increasing on each interval $[a_1 + \cdots + a_{s-1} + 1, a_1 + \cdots + a_s]$. Such elements are called $(a_1, \ldots, a_w)$-*unshuffles*; let $\mathrm{unsh}((a_1, \ldots, a_w) \subset \Sigma_n$ denote the set of all these unshuffles.

Now let

$$J_n^{(2)} := (P_n \times \Sigma_n)/ \sim,$$

with the relation $\sim$ defined as follows: for each face $(j_1, \ldots, j_w)$ of P_n and for each $p \in (j_1, \ldots, j_w)$ we identify

$$p\tau \times \sigma \sim p \times \tau\sigma,$$

where $\tau \in \mathrm{unsh}(a_1, \ldots, a_w)$ is the unshuffle corresponding to $(j_1, \ldots, j_w)$.

For higher $k > 2$ is the description of the relation in (2.4) more complicated, so we will not it give here and refer to [**?**] or [**?**, page 24] instead.

The collection $J^{(k)} = \{J_n^{(k)}\}_{n \geq 1}$ does not form an operad nor do the configuration spaces $F(\mathbb{R}^k) = \{F(\mathbb{R}^k, n)\}_{n \geq 1}$, where by $F(\mathbb{R}^k, n)$ we denote the configuration space of n distinct labeled points in the affine space $\mathbb{R}^k$ with the group Σ_n acting by permuting the labels. But it turns out that for the purpose of approximation the relevant structure is that of the degeneracies which correspond in an algebra to the reduction of relations by replacing one variable by a unit. This structure is formalized in the definition of a preoperad [**?**] (or coefficient system, cf. [**?**]).

Definition 2.4.2. *A (topological)* preoperad *is a functor* $\mathcal{O} : \mathbf{\Lambda}^{\mathrm{op}} \to \mathtt{Top}$, *where* $\mathbf{\Lambda}$ *is the category of (non-empty) finite sets and injective maps.*

It is clear that morphisms of the category $\mathbf{\Lambda}$ are generated by permutations and maps $d_i^n : \{1, \ldots, n\} \to \{1, \ldots, n+1\}$ defined, for $1 \leq i \leq n+1$, by

$$d_i^n(t) := \begin{cases} t, & \text{for } 1 \leq t < i \text{ and} \\ t+1, & \text{for } i \leq t \leq n. \end{cases} \tag{2.6}$$

It is clear from this description that a preoperad $\mathcal{O}$ is uniquely determined by a collection $\mathcal{O} = \{\mathcal{O}(n)\}_{n\geq 1}$ together with degeneracy maps $D_i^* := \mathcal{O}(d_i^n) : \mathcal{O}(n+1) \to \mathcal{O}(n)$, $i = 1, \ldots, n+1$, satisfying the appropriate relations. For a *pointed operad* $\mathcal{O}$, these degeneracy maps are given by *evaluation*:

$$D_i^*(t) := \gamma_{\mathcal{O}}(t; 1, \ldots, 1, *, 1, \ldots, 1) \quad (* \in \mathcal{O}(0) \text{ at the } i\text{th position})$$

where $t \in \mathcal{O}(n+1)$, $1 \in \mathcal{O}(1)$ is the unit of the operad, $* \in \mathcal{O}(0)$ is the constant and $1 \leq i \leq n+1$.

In the particular case of the operad $\mathcal{M}on$ for monoids with $\mathcal{M}on(n) = \Sigma_n$, $n \geq 1$, the degeneracies $D_i^* : \Sigma_{n+1} \to \Sigma_n$ are given by deleting i and its image $\sigma(i)$ for each permutation $\sigma \in \Sigma_{n+1}$. To be more precise,

$$D_i^*(\sigma) = e_{\sigma(i)}^n \circ \sigma \circ d_i^n,$$

where d_i was defined in (2.6) and

$$e_j^n : \{1, \ldots, j-1\} \cup \{j+1, \ldots, n+1\} \to \{1, \ldots, n\}$$

is, for $1 \leq j \leq n+1$, the unique order-preserving isomorphism. Degeneracies for the permutahedra are then induced by taking convex hulls of the maps D_i^* defined above. They can be shown to commute with the definition relation of (2.4) giving thus rise to a preoperad structure on the collection $J^{(k)} = \{J^{(k)}(n)\}_{n\geq 1}$.

Similarly, for the little cubes operad we have degeneracies $D_i^* : \mathcal{C}_k(n+1) \to \mathcal{C}_k(n)$ given by omitting the i-th little cube, and for the the configuration spaces, degeneracies $D_i^* : F(\mathbb{R}^k, n+1) \to F(\mathbb{R}^k, n)$ given by omitting the i-th point of the configuration.

In general, for any preoperad $\mathcal{O}$ and space $(X, *)$, we can form the space

$$\mathcal{O}X := (\coprod_{n\geq 1} \mathcal{O} \times_{\Sigma_n} X^n)/\sim,$$

where the equivalence relation is given by

$$\begin{aligned}\mathcal{O}(n) \times X^n \ni (t, x_1, \ldots, x_i = *, \ldots, x_n) \sim \\ \sim (D_i^*(t), x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in \mathcal{O}(n-1) \times X^{n-1},\end{aligned}$$

$1 \leq i \leq n$.

Now apply this construction using $J^{(k)}$ to define the *Milgram model* $J^{(k)}X$ of $\Omega^k S^k X$ for any space $(X, *)$. For $n = 1$, $J^{(1)}X$ is just the free monoid generated by the pointed space $(X, *)$, that is, the James' model for $\Omega S X$.

Using the preoperads $F(\mathbb{R}^k)$ or $\mathcal{C}_k X$ instead we get the *May models* $F(\mathbb{R}^k)X$ and $\mathcal{C}_k X$ of $\Omega^k S^k X$.

Theorem 2.4.3. *(Berger [?]) The configuration preoperad $F(\mathbb{R}^r)$, Milgram's preoperad $J^{(k)}$ and the little cubes operad $\mathcal{C}_k$ are homotopy equivalent as preoperads. For a well-pointed space $(X,*)$, the May models and the Milgram model are all (weakly?) homotopy equivalent.*

It then follows from Theorem **??** of [?] that all three models in Theorem 2.4.3 have homotopy type of the k-loop space $\Omega^k S^k X$.

2.5. Γ-spaces.

Version of May 6, 2000

Segal's' recognition principle and construction is similar but with several significant differences. In [?], he treats only infinite loop spaces (loop spectra in his terminology. In an unpublished manuscript, Bousfield [?] adapts Segal's approach to finitely iterated loop spaces. Instead of finding an operad which is sufficiently elastic to keep track of all the homotopies necessary, Segal uses a special discrete category Γ to define 'special' simplicial spaces. In the following definition, 2^T denotes the set of all subsets of T.

Definition 2.5.1. *Γ is the category whose objects are all finite sets and whose morphisms from S to T are maps $\theta : S \to 2^T$ such that $\theta(a)$ and $\theta(b)$ are disjoint if a and b are distinct.*

Let $\mathbf{n}$ denote the set $\{1, 2, \ldots, n\}$. The following definition of Γ spaces should be compared to that of a finitary algebraic theory of Lawvere [?].

Definition 2.5.2. *A* Γ-space *is a contravariant functor $\mathbf{A}$ from Γ to topological spaces such that*

(i) $\mathbf{A}(\mathbf{0})$ *is contractible, and*
(ii) for any n, the map $p_n : \mathbf{A}(\mathbf{n}) \to \mathbf{A}(\mathbf{1}) \times \cdots \times \mathbf{A}(\mathbf{1})$ induced by the n obvious maps $\mathbf{1} \to \mathbf{n}$ is a homotopy equivalence.

Any Γ-space can be regarded as a simplicial space (hence the terminology 'special' simplicial spaces) via the functor from the category Δ to Γ which takes the finite ordered set $[n] = \{0, 1, \cdots, n\}$ to $\mathbf{n}$ and a non-decreasing map $f : [p] \to [q]$ to $\theta : \mathbf{p} \to \mathbf{q}$ with $\theta(i) = \{j \in \mathbf{q} : f(i-1) < j \le f(i)\}$. Any simplicial space $\mathbf{A}$ has a realization as a topological space (in fact, there are several, all of the same homotopy type for reasonable spaces). Using an appropriate realization, Segal constructs a 'classifying space' $B\mathbf{A}$ for any Γ-space $\mathbf{A}$ which is again a Γ-space. Also, any Γ-space has an H-space structure on $\mathbf{A}(1)$ given by the map

$$\mathbf{A}(\mathbf{1}) \times \mathbf{A}(\mathbf{1}) \xrightarrow{p_2^{-1}} \mathbf{A}(\mathbf{2}) \xrightarrow{\mathbf{A}(\nu)} \mathbf{A}(\mathbf{1}),$$

where p_2^{-1} is an arbitrary homotopy inverse of $p_2 : \mathbf{A}(\mathbf{2}) \to \mathbf{A}(\mathbf{1}) \times \mathbf{A}(\mathbf{1})$ and $\nu : \mathbf{1} \to \mathbf{2}$ is the map sending $1 \in \mathbf{1}$ to $\{1, 2\} \subset 2^{\mathbf{2}}$.

Proposition 2.5.3. *$\mathbf{A}(1)$ has the homotopy type of $\Omega B\mathbf{A}(1)$ if and only if the H-space structure on $\mathbf{A}(1)$ has a homotopy inverse.*

He observes that $B\mathbf{A}(1)$ in turn always has a homotopy inverse for its H-space structure and so $\mathbf{A}(1)$ is then an infinite loop space.

As for approximation, Segal considers $\coprod B\Sigma_n$ (where $B\Sigma_n$ is the classifying space of the symmetric group) as $B\Sigma(\mathbf{1})$ of a Γ-space $B\Sigma$ induced from the category of finite sets with disjoint union as monoidal structure. Similarly, for a space X, he considers a Γ-space $B\Sigma_X$ with $B\Sigma_X(\mathbf{1}) = \coprod(E\Sigma_n \times X^n)/\Sigma_n$ (where $E\Sigma_n$ is the total space of a universal principal Σ_n-bundle).

Theorem 2.5.4. *(Barratt-Priddy-Quillen) As an infinite loop space, $B(B\Sigma)$ is homotopy equivalent to $\Omega^\infty S^\infty$, the 'sphere spectrum'.*

Theorem 2.5.5. *As an infinite loop space, $B(B\Sigma_X)$ is homotopy equivalent to $\Omega^\infty S^\infty X_+$, where X_+ denotes the disjoint union of X with a point.*

2.6. Homology operations.

Version of May 13, 2000

In this section we present a unifying operadic approach to primary (co)homology operations, due to P. May [89], and a brief survey of multi-variable higher order operations. We will respect the original sign and degree conventions, though in some cases they might be different from conventions used today.

We first review the Steenrod algebra in terms of higher homotopy commutativity. Let $K := C^*(X)$ be the complex of singular cochains on a topological space X. The classical cup product $\smile\colon K\otimes K \to K$ is a (strictly) associative multiplication. It is commutative up to a homotopy, which is traditionally denoted by $\smile_1$:

$$u \smile v - (-1)^{|u||v|} v \smile u = \delta(u \smile_1 v) + \delta u \smile_1 v + u \smile_1 \delta v. \tag{2.7}$$

The homotopy $\smile\colon K\otimes K \to K$ is homotopy anticommutative and there exists a hierarchy $\{\smile_i\}_{i\geq 1}$ of homotopies such that

$$(-1)^i u \smile_i v - (-1)^{|u||v|} v \smile_i u = \delta(u \smile_{i+1} v) + (-1)^i \delta u \smile_{i+1} v + (-1)^{i+|u|} u \smile_{i+1} \delta v.$$

Using the concept of *graded* commutativity **??**, we may read the above equation as saying that the degree $-i$ map $\smile_i$ is graded commutative up to a homotopy $\smile_{i+1}$. Steenrod squares $\mathrm{Sq}^i : H^p(X;\mathbb{Z}_2) \to H^{p+i}(X;\mathbb{Z}_2)$ in mod 2 cohomology are then obtained as $\mathrm{Sq}^i([u]) := [u \smile_{p-i} u]$.

The above can be rewritten as follows. Let $\mathbb{Z}[\mathbb{Z}_2]$ be the group ring of the abelian group $\mathbb{Z}_2$ and let $W = (W, d)$ be the 'standard' $\mathbb{Z}[\mathbb{Z}_2]$-free resolution of $\mathbb{Z}$ taken with the opposite grading, i.e. $W_i :=$ the $\mathbb{Z}[\mathbb{Z}_2]$-free module with one generator e_i of degree $i \leq 0$, and the differential d acting as

$$de_i := e_{i+1} + (-1)^i T e_{i+1}, \quad T \text{ the generator of } \mathbb{Z}_2.$$

The reason why we take W with the opposite grading is so that the differential of the resolution has the same degree as the differential in the cochain complex $C^*(X)$, which is $+1$. Another possibility would be to flip the grading of $C^*(X)$, the convention used in [89, par. 5]. It is easy to verify that the map $\theta : W \otimes K^{\otimes 2} \to K$ defined by

$$\theta(e_{-i} \otimes u \otimes v) = (-1)^{i(i-1)/2} u \smile_i v \ \text{ and } \ \theta(Te_{-i} \otimes u \otimes v) = (-1)^{i(i-1)/2 + |u||v|} v \smile_i u, \tag{2.8}$$

is $\mathbb{Z}_2$-equivariant with $\theta(e_0 \otimes u \otimes v) = u \smile v$. An elementary exposition of these facts can be found in [94, Section 2].

More generally, it follows from an acyclic models argument that, for an arbitrary prime number $r > 2$ and a $\mathbb{Z}[\mathbb{Z}_r]$-free resolution W of $\mathbb{Z}$ such that W_0 is generated by e_0, there exists a $\mathbb{Z}_r$-equivariant map $\theta : W \otimes K^{\otimes r} \to K$ of chain complexes such that $\theta|_{\{e_0\} \otimes K^{\otimes r}}$ is the iterated cup product. This map gives rise to mod r reduced powers $P^i : H^p(X; \mathbb{Z}_r) \to H^{p+2i(r-1)}(X; \mathbb{Z}_r)$, see [119, VII] for details.

A similar situation arises when we take $K := C_*(Y)$, the singular chain complex of an $(n+1)$-fold loop space. Then, as we know **??**, the space Y admits an action of the little $(n+1)$-cubes operad $\mathcal{C}_{n+1}$ which has the property that $\mathcal{C}_{n+1}(r)$ is a Σ_r-free, $(n-1)$-connected space.

Let W be the same $\mathbb{Z}[\mathbb{Z}_r]$-free resolution as above, but this time with the standard, positive grading. The resolution W will not act on the chains $C_*(Y)$, but $W^{(n)}$, the n-skeleton of W, will. The $\mathbb{Z}[\mathbb{Z}_r]$-freeness of $W^{(n)}$ together with the $(n-1)$-acyclicity of $C_*(\mathcal{C}_{n+1}(r))$ give a $\mathbb{Z}_r$-equivariant map $W^{(n)}_* \to C_*(\mathcal{C}_{n+1}(r))$. This map, combined with the action $\mathcal{C}_{n+1}(r) \times Y^{\times r} \to Y$, induces a $\mathbb{Z}_r$-equivariant map $\theta : W^{(n)}_* \otimes K^{\otimes r} \to K$ with the property that $\theta|_{\{e_0\} \otimes K^{\otimes r}}$ is chain homotopic to the loop space multiplication. By the same mechanism as above, the map θ gives rise to Kudo-Araki operations, introduced in [**?**]

$$Q^i_{(2)} : H_p(Y; \mathbb{Z}_2) \to H_{p+i}(Y; \mathbb{Z}_2), \text{ for } p \leq i \leq p+n,$$

and, if $r > 2$ is an odd prime, Dyer-Lashof operations, introduced in [24]:

$$Q^i_{(r)} : H_p(Y; \mathbb{Z}_r) \to H_{p+2i(r-1)}(Y; \mathbb{Z}_r), \text{ for } p(r-1) \leq 2i(r-1) < p(r-1) + n.$$

Let Y be an $(n+1)$-fold loop space as above and consider again the action $\theta : W^{(n)} \otimes K^{\otimes 2} \to K$. As in (2.8), the formula $u \smile_i v := (-1)^{i(i-1)/2} \theta(e_i \otimes u \otimes v)$ defines, for $0 \leq i \leq n$, a family of bilinear degree i maps such that $\smile_i$ is, for $0 \leq i \leq n-1$, graded commutative up to the homotopy $\smile_{i+1}$. The obstruction cycle to the graded homotopy commutativity of $\smile_n$ (taken with appropriate signs) induces, for any coefficient ring Λ, the so-called Browder operation [13]:

$$\lambda_n : H_p(Y;\Lambda)\otimes H_q(Y;\Lambda) \to H_{p+q+n}(Y;\Lambda).$$

The importance of the operation λ_n is related to the property that $\lambda_n = 0$ if Y is an (n+2)-fold loop space, thus it forms an obstruction for Y to be a $(n+2)$-fold loop space. The map λ_n is a degree n map which satisfies the following form of the graded Jacobi identity

$$(-1)^{(a+n)(c+n)}\lambda_n(x,\lambda_n(y,z))+(-1)^{(b+n)(c+n)}\lambda_n(y,\lambda_n(z,x))+(-1)^{(c+n)(b+n)}\lambda_n(z,\lambda_n(x,y)) = 0,$$

for $x \in H_a(Y;\Lambda)$, $y \in H_b(Y;\Lambda)$ and $z \in H_c(Y;\Lambda)$, and which is symmetric in the sense that

$$\lambda_n(x,y) = (-1)^{ab+1+n(a+b+1)}\lambda_n(y,x), \text{ for } x \in H_a(Y;\Lambda) \text{ and } y \in H_b(Y;\Lambda).$$

The sign convention follows the original work of Browder. Today, one would replace $\lambda_n(x,y)$ by $(-1)^{|x|}\lambda_n(x,y)$, this new λ_n would be then a graded antisymmetric degree n-Lie bracket, see **??**.

Let us remark that W. Browder introduced his operations for a quite large class of topological spaces, which he called H_n-spaces. A H_n-space is, by definition, a topological space Y equipped with an '$\mathbb{S}^n$-action,' which is a $\mathbb{Z}_2$-equivariant map

$$\phi : \mathbb{S}^n \times Y^{\times 2} \longrightarrow Y,$$

with $\mathbb{Z}_2$ acting on the sphere $\mathbb{S}^n$ via the antipodal map. It is easy to see that any $(n+1)$-fold loop space is an H_n-space.

Common features of the above constructions are summarized in the following definition due to May [89]. For an integer r, a commutative ring Λ, a subgroup $\pi \subset \Sigma_r$ of the symmetric group Σ_r, and for a $\Lambda[\pi]$-free resolution W of Λ with $W_0 = \Lambda e_0$, he introduced a category $\mathcal{C}(\pi,n,\Lambda)$, whose objects are pairs (K,θ) consisting of a homotopy associative (but not strongly homotopy associative in the sense of **??**) differential Λ-algebra K and a $\Lambda[\pi]$-equivariant morphism $\theta : W^{(n)}\otimes K^{\otimes r} \to K$ of differential complexes, where $W^{(n)}$ denotes the n-skeleton of W. The map θ has to satisfy several conditions, the most important being the requirement that the restriction of θ to $\{e_0\}\otimes K^{\otimes r}$ is Λ-homotopic to the iterated product $K^{\otimes r} \to K$.

If π is the cyclic group of prime order r, we write simply $\mathcal{C}(r,n)$ instead of $\mathcal{C}(\pi,n,\mathbb{Z}_r)$. Suppose that $(K,\theta) \in \mathcal{C}(r,n)$. If $r = 2$, then θ induces operations

$$P_i : H_p(K) \to H_{p+i}(K), \text{ for } i \leq p+n.$$

Similarly, for an odd prime $r > 2$, there are the operations

$$P_i : H_p(K) \to H_{p+2i(r-1)}(K),$$

for $2i(r-1) \leq p(r-1)+n$, and

$$\beta P_i : H_p(K) \to H_{p+2i(r-1)-1}(K),$$

for $2i(r-1) \leq p(r-1)+n+1$. In the general case, βP_i need not be the composition of the Bockstein with P_i as suggested by the notation. This is, however, true for so-called r-reduced objects $(K,\theta) \in \mathcal{C}(r,n)$. By definition, (K,θ) is r-reduced if it is obtained as mod r reduction of some $(\widetilde{K},\widetilde{\theta}) \in \mathcal{C}(\pi,n,\mathbb{Z})$, where $\widetilde{K}$ is a flat $\mathbb{Z}$-module. Observe that both examples discussed above are r-reduced.

Let us give another, purely algebraic, example. For an associative Λ-algebra A, let $CH^p(A,A) := \mathrm{Hom}_\Lambda(A^{\otimes p}, A)$ be the complex of Hochschild cochains, with Hochschild differential δ **??**. In the classical paper [31], M. Gerstenhaber considered the (obviously associative) cup product $\smile$: $CH^p(A;A) \otimes CH^q(A;A) \to CH^{p+q}(A;A)$:

$$(f \smile g)(a_1 \otimes \cdots \otimes a_p \otimes b_1 \otimes \cdots \otimes b_q) := f(a_1 \otimes \cdots \otimes a_p) g(b_1 \otimes \cdots \otimes b_q).$$

The cup product induces an associative product (which we denote by the same symbol) in the Hochschild cohomology $HH^*(A;A)$. Some indications based on the relation to deformation theory of associative algebras suggested that this product must be commutative, though the definition did not look commutative at all!

To prove this, M. Gerstenhaber introduced, for $f \in CH^p(A;A)$ and $g \in CH^q(A;A)$, the composition product $f \circ g \in CH^{p+q-1}(A;A)$ by

$$(f \circ g)(a_1 \otimes \cdots \otimes a_{p+q-1}) := \sum_{0 \leq i \leq p} (-1)^{iq} f(a_1 \otimes \cdots \otimes a_{i-1} \otimes g(a_i \otimes \cdots \otimes a_{i+q}) \otimes a_{q+i+1} \otimes \cdots \otimes a_{p+q-1}),$$

and proved the formula

$$f \circ \delta g - \delta(f \circ g) + (-1)^{q-1} \delta f \circ g = (-1)^{q-1}[g \smile f - (-1)^{pq} f \smile g]$$

which means that the chain-level cup product is homotopy commutative. A harmless modification of signs, $f \smile_1 g := (-1)^{pq+q} f \circ g$, gives a $\smile_1$-product satisfying (2.7). The corresponding Browder operation λ_1 defines the bracket product $[x,y] := (-1)^{|x|}\lambda_1(x,y)$ in Hochschild cohomology. The triple $(HH^*(A;A), \smile, [-,-])$ (again with a small sign modification) is an example of algebraic structure which is called today a Gerstenhaber or G-algebra, see **??**.

We saw that Hochschild complex exhibits some properties of 2-fold loop spaces. This lead P. Deligne [22] to conjecture that the complex $CH^*(A;A)$ admits an action of some operad, chain homotopy equivalent to the operad of singular chains on the little 2-cubes operad $\mathcal{C}_2$. Such an action was indeed very explicitly constructed in a recent work of M. Gerstenhaber and A.A. Voronov [33], see Section **??**.

Heuristically, the existence of a 'W-action' $\theta : W \otimes K^{\otimes r} \to K$ means that the multiplication in K is homotopy commutative (or, if $r > 2$, that its iterates are homotopy symmetric), with a hierarchy of higher coherent homotopies. We saw that the 'W-action' was in some concrete examples induced from an action of an acyclic or at least highly connected operad, which meant that the multiplication in K was homotopically very close to a commutative and associative one, again with a coherent system of higher homotopies.

It is thus a very deep fact that also the first example of this section, the example of reduced chain complex of a topological space, admits an action of an acyclic operad.

One such an action was constructed by V.A. Smirnov [111] (and independently also by V.A. Hinich and V.V. Schechtman in [49]) who proved that the 'Eilenberg-Zilber' operad $\mathcal{E}$ of endomorphisms of the reduced chain complex of the simplex acts on $C_*(X)$. Notice, however, that the Eilenberg-Zilber operad is not concentrated in positive degrees and it is not Σ-free.

Another, closely related, construction belongs to J. Smith [113]. He constructed an operad $R = \{R(n)\}_{n\geq 0}$ with the property that $R(n)$ is the bar resolution of $\mathbb{Z}$ over the symmetric group ring $\mathbb{Z}[\Sigma_n]$ and proved that also this operad acts naturally on the singular chain complex $C_*(X)$. Both actions contain very strong information about the space X itself, for instance, under some mild assumptions they fix the weak homotopy type of X. For a p-adic version, see M.A. Mandell [78].

The operations considered by P. May for which operads are so useful are operations derived from the higher homotopy commutativity embodied in the Eilenberg-Zilber theorem, so the action of the symmetric groups plays a key role. On the other hand, the non-symmetric A_∞-operad $\underline{\mathcal{A}ss}_\infty$ is relevant to the higher order operations of several variables introduced by Uehara and Massey [**?**, **?**], even though the Alexander-Whitney map provides a strictly associative multiplication at the cochain level.

The definition of Massey products in the homology of a loop space [**?**] makes essential use of the A_∞-structure on the chains of an A_∞-space. For example, the definition of the Massey triple product $<u,v,w>$ uses the associating homotopy m_3. For $u,v,w \in H(A)$ such that $uv = 0 = vw$, we can choose representatives $\bar{u} \in u, \bar{v} \in v, \bar{w} \in w$ and chains $a,b \in A$ such that $\bar{u}\bar{v} = da$ and $\bar{v}\bar{w} = db$. Then $(-1)^{|u|}\bar{u}b - a\bar{w}$ is not necessarily a cycle, but $(-1)^{|u|}\bar{u}b - a\bar{w} + m_3(\bar{u}\otimes\bar{v}\otimes\bar{w})$ is and represents $<u,v,w>$, which is defined as the coset of $H_*(A)$ modulo the indeterminacy $uH_*(A) + H_*(A)w$ to accommodate the choices made.

There is another way of choosing a representative which we illustrate in the original setting of associative cochain algebras. As proved in **??**, an associative algebra A admits the structure of an A_∞-algebra for which the original homotopy equivalence is an A_∞-map. In particular, the cohomology $H(A)$ with $d = 0$ is such a complex if we have a field as ground ring. Since $d = 0$, the algebra $H(A)$ is strictly associative, but the induced A_∞-structure maps $m_i : H(A)^{\otimes i} \to H(A)$, $i \geq 3$ need not be trivial. For example, $m_3 : H(A)^{\otimes 3} \to H(A)$ may be a non-trivial homotopy between the unique triple product and itself. The operation

m_3 is related to the Massey triple product as follows: if $uv = vw = 0$, then $m_3(u, v, w)$ represents $<u, v, w>$ in $H(A)$ modulo $uH(A) + H(A)w$.

The usefulness of the Massey product is well illustrated in two important examples: the Massey triple product which detects the linking of the Borromean rings [?, ?] and the Massey triple products in $H^*(Sp(5)/SU(5))$ which distinguish the homotopy type of $Sp(5)/SU(5)$ from that of the connected sum $(S^6 \times S^{25})\#(S^{10} \times S^{21})$ [?, ?] which has the same cohomology algebra.

If the A_∞-structure on $H(A)$ induced from that on A is trivial, A is called *formal*, a term first introduced in rational homotopy theory. The concept of *formality* plays an important role in many recent developments, especially in relation to deformation quantization [?] which uses the L_∞-operad **??**, providing the Lie analog of A_∞-structure **??**, see also Section **??**. The Lie analogs of Massey products are known as Lie-Massey brackets or just Massey brackets [104].

2.7. The linear isometries operad and infinite loop spaces.

Version of May 15, 2000

Many infinite loop spaces are recognized in terms of an operad, called the *linear isometries operad* and denoted $\mathcal{L}i$, which more recently has played a major role in a streamlined approach to the stable homotopy category [?]. Following Boardman and Vogt [10], we describe this operad as follows.

Let `LI` denote the symmetric monoidal category of real inner-product spaces of countable dimension and linear isometric maps. Let us point out that by a linear isometric map we mean just a linear isometric *embedding*, not necessarily an isomorphism.

Topologize objects of `LI` by the direct limit topology of all the finite dimensional subspaces, which are given the usual metric topology. Topologize the morphism sets with the compactly generated function space topology. The monoidal structure is that of direct sum with the canonical identifications of vector spaces for associativity, commutativity and $\mathbb{R}$ as unit for tensor products.

There is a special object in the category `LI`, the direct limit

$$\mathbb{R}^\infty := \varinjlim \mathbb{R}^n.$$

The basic property of $\mathbb{R}^\infty$ is formulated in the following lemma, whose proof we take almost literally from [10, pages 207–208].

Lemma 2.7.1. *The space* $\mathtt{LI}(V, \mathbb{R}^\infty)$ *is contractible for any vector space* $V \in \mathrm{Ob}(\mathtt{LI})$.

Proof. A contracting homotopy is easy to construct, its construction being somewhat analogous to the 'Rota swindle.' Let $\{e_1, e_2, \ldots\}$ be an orthonormal basis for $\mathbb{R}^\infty$. Let $h : \mathbb{R}^\infty \to \mathbb{R}^\infty \oplus \mathbb{R}^\infty$ be the isometry

$$h(e_{2n}) = (e_n, 0) \ \text{ and } \ h(e_{2n-1}) = (0, e_n), \ n \geq 1.$$

By applying the Gram-Schmidt orthogonalization process to the homotopy given by

$$g_t(e_n) := (1-t)e_n + te_{2n}, \ n \geq 1, \ t \in [0,1],$$

we obtain a homotopy through isometries from $\mathbb{1}_{\mathbb{R}^\infty}$ to $g := g_1$, given by $g(e_n) := e_{2n}$.

Similarly, construct a homotopy between the two axial inclusions $i_1, i_2 : V \to V \oplus V$ by applying Gram-Schmidt process to the homotopy

$$(1-t)i_i + ti_2, \ t \in [0,1].$$

Let $k : V \to \mathbb{R}^\infty$ be a fixed isometry. Let us compose the above homotopies to define a contraction of $\mathcal{L}i(V, \mathbb{R}^\infty)$ into the point $h^{-1} \circ (k \oplus k) \circ i_2 \in \mathcal{L}i(V, \mathbb{R}^\infty)$.

For any isometry $f : V \to \mathbb{R}^\infty$, the contraction runs from $f = h^{-1} \circ h \circ f$ to

$$h^{-1} \circ h \circ g \circ f = h^{-1} \circ i_1 \circ f = h^{-1} \circ (f \oplus k) \circ i_1$$

(using the homotopy between $\mathbb{1}_{\mathbb{R}^\infty}$ to g) and thence to $h^{-1} \circ (f \oplus k) \circ i_2 = h^{-1} \circ (k \oplus k) \circ i_2$ (using the homotopy between i_1 and i_2). ■

The 'classical' infinite loop spaces can be described in terms of symmetric monoidal functors $\mathtt{LI} \to \mathtt{Top}$. The category $\mathtt{LI}$ determines the *linear isometry operad*, denoted $\mathcal{L}i$.

Definition 2.7.2. *The operad* $\mathcal{L}i = \{\mathcal{L}i(n)\}_{n \geq 1}$ *is given by*

$$\mathcal{L}i(n) := \mathtt{LI}(\oplus^n \mathbb{R}^\infty, \mathbb{R}^\infty), \ n \geq 1.$$

Strictly speaking, the n-fold direct sum is given by a fixed choice of iteration. The symmetric groups act by permuting the summands in $\oplus^n \mathbb{R}^\infty$; since the maps in $\mathcal{L}i(n)$ are imbeddings, the induced action is free.

Theorem 2.7.3. *The operad* $\mathcal{L}i$ *is an* E_∞*-operad. Any continuous symmetric monoidal functor* $\mathtt{F} : \mathtt{LI} \to \mathtt{Top}$ *induces, up to homotopy equivalence, an infinite loop space structure on* $\mathtt{F}(\mathbb{R}^\infty)$.

Proof. The first part of the theorem follows immediately from Lemma 2.7.1. For the second part, let $X := \mathtt{F}(\mathbb{R}^\infty)$. Application of a continuous symmetric monoidal functor $\mathtt{F} : \mathtt{LI} \to \mathtt{Top}$ induces, for $n \geq 1$, maps

$$\mathcal{L}i(n) = \mathtt{LI}(\oplus^n \mathbb{R}^\infty, \mathbb{R}^\infty) \longrightarrow \mathcal{E}nd(X)(n) = Map(X^n, X)$$

which form a map of operads: $\mathcal{L}i \to \mathcal{E}nd(X)$. Thus $\mathtt{F}(\mathbb{R}^\infty)$ is an $\mathcal{L}i$-space and hence it has the homotopy type of an infinite loop space, by Theorem **??**. ■

Applications of Theorem 2.7.3 abound. The classical ones are given in the following examples.

Example 2.7.4. Let $\mathtt{F} : \mathtt{LI} \to \mathtt{Top}$ be given by $\mathtt{F}(V) := O(V)$, the orthogonal group of V, so $\mathtt{F}(\mathbb{R}^\infty) = O(\mathbb{R}^\infty)$ is the stable orthogonal group, which is thus an infinite loop space. We may think of the multiplication as being induced by

$$A \in O(p), B \in O(q) \longmapsto A \oplus B \in O(p+q)$$

where, in terms of matrices,

$$A \oplus B := \left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right).$$

Observe that the infinite loop space structure above does not use the group structure of the orthogonal group.

We can apply the ordinary classifying space functor B so that $\mathtt{F}(V) := BO(V)$. This choice also gives a monoidal functor, as follows from the isomorphism

$$B(G \times H) \cong BG \times BH$$

for any two compact groups G and H. Thus the classifying space $BO := BO(\mathbb{R}^\infty)$ for real K-theory is again an infinite loop space with respect to the multiplication given by the classifying map for the Whitney sum of bundles.

The analogous results for U and BU (respectively Sp and BSp) follow immediately by considering $\mathtt{F}(V) = U(V \otimes \mathbb{C})$ (respectively $\mathtt{F}(V) = U(V \otimes \mathbb{H})$ where $\mathbb{H}$ denotes the quaternions).

Example 2.7.5. Only slightly more subtle is $\mathtt{F}(V) := \mathcal{H}(V \cup \infty)$, the space of homotopy self-equivalences of the sphere obtained as the one-point compactification of V. For $\mathtt{F}(V) := Homeo(V)$, the space of homeomorphisms of V to itself, the topology is considerably more subtle; Boardman and Vogt resort to replacing $Homeo(V)$ by the weakly equivalent realization of its singular complex.

The maps $V \to V \otimes \mathbb{C} \to V \otimes \mathbb{H}$ induce infinite loop space maps as do the forgetful maps $V \otimes \mathbb{C} \to V \otimes \mathbb{R}^2$ and $V \otimes \mathbb{H} \to V \otimes \mathbb{R}^4$ as well as the maps $O(V) \to Homeo(V) \to \mathcal{H}(V \cup \infty)$ and all the respective maps of classifying spaces.

Example 2.7.6. The tensor product of matrices

$$\otimes : O(p) \times O(q) \to O(pq), \quad A, B \mapsto A \otimes B,$$

induces a different structure on the infinite orthogonal group $O(\mathbb{R}^\infty)$. This, and the corresponding structure on the classyfying space BO, can again be shown to be an infinite loop space structure, but this time using Segal's approach **??** and not that of Boardman and Vogt, as they point out. The analogous results are true also for U, Sp, etc.

2.7. W-construction.

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Another major contribution of Boardman and Vogt [10, 9] is their W-construction, which was their main tool in the development of homotopy invariant structures. Here we will present the essential ideas of the W-construction, its properties and applications. A modern sophisticated treatment has recently been given by Vogt [131], but ours will be closer to the original, which we find more transparent as to the basic idea.

Theorem 2.7.1. *The* W-construction *is a functor* W : `Op` $\to$ `Op` *which, from any (topological) operad* $\mathcal{O}$*, produces an operad* $W\mathcal{O}$ *and an augmentation* $\varepsilon : W\mathcal{O} \to \mathcal{O}$ *which is a cofibrant resolution of* $\mathcal{O}$*.*

'Cofibrant' refers to an appropriate Closed Model Category structure, which we take up in Section **??**. One of the implications of the cofibrancy is that, given a diagram of operads

$$\begin{tikzcd} & \mathcal{B} \arrow[d, "g"] \\ W\mathcal{A} \arrow[ur, dotted, "\overline{f}"] \arrow[r, "f"] & \mathcal{C} \end{tikzcd}$$

with g a homotopy equivalence of operads, there exists a lift $\overline{f} : W\mathcal{A} \to \mathcal{A}$ such that $g \circ \bar{f}$ and f are homotopic *as maps of operads.*

For a detailed proof, see [10, Proposition II.3.6, Theorem II.3.17]. Essential ideas of the construction $W\mathcal{O}$ can be demonstrated on the operad they call *the operad of trees*, but which we prefer to call the *operad of metric trees*.

Definition 2.7.2. *A* metric tree *is a tree in the sense of Section* ?? *(i.e. with one root and a finite number of leaves) together with a* length function $h : \mathrm{Edg}(T) \to [0,1]$.

For a given tree T, the set of metrics $\mathrm{Met}(T)$ on T can be realized as a cube I^k, where the dimension k is the number of internal edges of the tree.

Recall that for an edge e of T, T/e denotes the tree obtained from T by shrinking e to a point. For any length function $h : \mathrm{Edg}(T/e) \to [0,1]$, let $s_e(h) : \mathrm{Edg}(T) \to [0,1]$ be the obvious function which is 0 on e and agrees with h on the other edges. So we have defined, for any $e \in \mathrm{Edg}(T)$, a 'degeneracy' $s_e : \mathrm{Met}(T/e) \to \mathrm{Met}(T)$.

Now define, for a given $n \geq 1$, the *space of metric trees* $\mathcal{T}m(n)$ of arity n to be the polytope formed from the union of all metric trees of arity n by identifying $\mathrm{Met}(T/e)$ as a cube with the corresponding face of the cube $\mathrm{Met}(T)$ or, formally,

$$\mathrm{Met}(T) \ni s_e(h) \sim h \in \mathrm{Met}(T/e),$$

for any $T \in \mathtt{Tree}_n$ and $e \in \mathrm{Edg}(T)$.

For metric trees, the grafting operation should be considered as identifying the root edge with the appropriate leaf edge and assigning the length of this new internal edge to be 1. This grafting induces (in fact, piecewise cubical) embeddings

$$\circ_i : \mathcal{T}m(r) \times \mathcal{T}m(s) \to \mathcal{T}m(r+s-1),\ \ r,s \geq 1,$$

giving $\mathcal{T}m = \{\mathcal{T}m(n)\}_{n\geq 1}$ the structure of a topological operad. The W-construction on an operad $\mathcal{O}$ is then a generalization of the operad $\mathcal{T}m$ in the sense that we consider metric trees with vertices colored by elements of the operad $\mathcal{O}$. A precise definition is given below.

The original construction [10] described $W\mathcal{O}$ as a quotient of the space of $\mathcal{O}$-colored metric trees by two types of relations, the first type used (in our terminology) the pseudo-operad structure of $\mathcal{O}$ only; the second, more complicated one involved the unit of $\mathcal{O}$. We will avoid introducing the second type of relation by assuming that

$$\mathcal{O}(1) = * \text{ (the one-point space);} \tag{2.9}$$

these operads are general enough for our purposes. Recall ?? that the functor $(-)^+ : \mathtt{Op} \to \Psi\mathtt{Op}$ defined by

$$\mathcal{O}^+(n) := \begin{cases} \emptyset, & \text{for } n = 1, \text{ and} \\ \mathcal{O}(n), & \text{for } n \geq 2, \end{cases}$$

induces an equivalence of the category of (topological) operads with $\mathcal{O}(1) = *$ and the category of pseudo-operads $\mathcal{P}$ with $\mathcal{P}(1) = \emptyset$, the inverse functor given by formally adjoining the unit.

So we start with an operad $\mathcal{O}$ satisfying (2.9), then consider the pseudo-operad $\mathcal{O}^+$, construct a pseudo-operad $W\mathcal{O}^+$, and finally define the operad $W\mathcal{O}$ by adjoining the unit to $W\mathcal{O}^+$. Recall **??** that

$$\mathcal{O}^+(T) = \bigsqcup_{v \in \mathrm{Vert}(T)} \mathcal{O}^+(In(v))$$

denotes the set of all $\mathcal{O}^+$-colorings of the tree T. Recall also that the pseudo-operad structure on $\mathcal{O}^+$ induces, for each $e \in \mathrm{Edg}(T)$, a map

$$\gamma_e : \mathcal{O}^+(T) \to \mathcal{O}^+(T/e).$$

Now let $W\mathcal{O}^+(1) := \emptyset$ and, for $n \geq 2$, define $W\mathcal{O}^+(n)$ as the quotient of

$$\bigsqcup_{T \in \mathtt{Tree}_n} \mathrm{Met}(T) \times \mathcal{O}^+(T) \tag{2.10}$$

by the relation:

$$\mathrm{Met}(T) \times \mathcal{O}^+(T) \ni (s_e(h), f) \sim (h, \gamma_e(f)) \in \mathrm{Met}(T/e) \times \mathcal{O}^+(T/e),$$

where $T \in \mathtt{Tree}_n$ and $e \in \mathrm{Edg}(T)$. The quotient is topologized using the topology on $\mathcal{O}$ together with that of the cubes. The pseudo-operad structure is given as follows. Observe that (2.10) is the union of products of ingredients making up the operad $\mathcal{T}m$ (the $\mathrm{Met}(-)$-factor) and those making up the free pseudo-operad (the $\mathcal{O}^+(-)$-factor). The pseudo-operad structure on $W\mathcal{O}^+$ is then induced diagonally by these two pseudo-operad structures. Finally, the operad $W\mathcal{O}$ is obtained by adjoining the unit $1 \in W\mathcal{O}(1) = *$ to $W\mathcal{O}^+$.

The augmentation $\varepsilon : W\mathcal{O} \to \mathcal{O}$ has components $\varepsilon(n) : W\mathcal{O}(n) \to \mathcal{O}(n)$ induced by the composition

$$\mathrm{Met}(T) \times \mathcal{O}^+(T) \xrightarrow{\text{proj.}} \mathcal{O}^+(T) \xrightarrow{\gamma_T} \mathcal{O}^+(n) = \mathcal{O}(n),\ T \in \mathtt{Tree}_n,\ n \geq 2,$$

where γ_T denotes the composition along $T \in \mathtt{Tree}_n$. Informally, this means that ε is defined by shrinking every edge (length) to 0 and correspondingly composing the operad elements associated to the vertices at the ends of the edge.

As a matter of fact, the collection $\mathcal{O}$ is a subcollection (but !not! a suboperad) of $W\mathcal{O}$ under the natural inclusion $\iota : \mathcal{O} \to W\mathcal{O}$ given by

$$\iota(n)(o) := h_{t(n)} \times o \in \mathrm{Met}(t(n)) \times \mathcal{O}^+(n),$$

where $h_{t(n)}$ is the unique metric on the corolla $t(n)$ with n input leaves, $n \geq 2$. Metric contraction of the edges induces a deformation contraction of $W\mathcal{O}$ to $\mathcal{O}$.

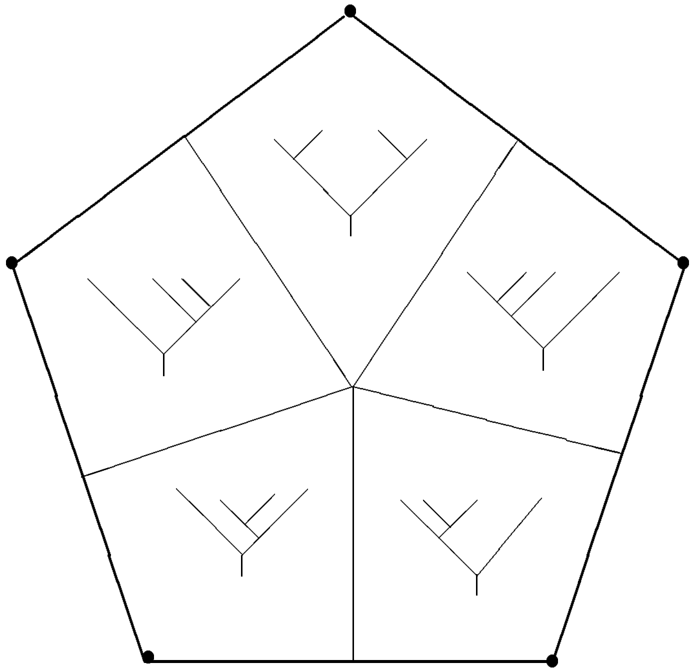


Figure 2.5: A portrait of CK_4. CK_4 is the pentagon decomposed into five cubes. The cubes are indexed by binary trees $T \in \mathtt{Tree}_4$.

Example 2.7.3. The operad $\mathcal{C}mon$ for commutative associative topological monoids (not to be mistaken with the operad $\mathcal{C}omm$ for commutative associative algebras) is given by $\mathcal{C}mon(n) = *$, for $n \geq 1$. The operad $\mathcal{T}m$ of metric trees is then exactly the W-construction on $\mathcal{C}mon$; $\mathcal{T}m = W\mathcal{C}mon$.

The W-construction applied to the operad $\mathcal{M}on$ for associative monoids decomposes as

$$W\mathcal{M}on(n) = CK_n \times \Sigma_n, \ n \geq 1,$$

where CK_n is a certain cubical decomposition of the associahedron K_n. A portrait of CK_4 is given in Figure 2.5.

Chapter 3

Algebra

3.1. The cobar complex of an operad.

In order to study the homological algebra of operads, in particular, homotopy algebra structures and the Koszul property, we need to extend our base category from $\mathcal{C} = \mathtt{Mod}_{\mathbf{k}}$, which was the main example in the previous two sections, to the category dg-$\mathtt{Mod}_{\mathbf{k}}$ of differential graded $\mathbf{k}$-modules.

In the following sections we will define several dg-complexes. The cobar complex $\mathbf{C}(A)$ and the dual dg-operad $\mathbf{D}(A)$ are more or less the same, up to operadic suspension (defined below). The Koszul complex $\mathbf{K}(A)$ is important because it occurs in the cochain complex of a free A-algebra which plays a role in a characterization of Koszulness of the operad A, which is defined below. The categorical cobar complex $\mathbf{N}(A)$ is the middle link in a quasi-isomorphism between $\mathbf{C}(A)$ and $\mathbf{K}(A)$ in sense that there are two spectral sequences on $\mathbf{N}(A)$, one converging to $\mathbf{C}(A)$, the second one converging to $\mathbf{K}(A)$.

The objects (V^*, d) of dg-$\mathtt{Mod}_{\mathbf{k}}$ are cochain complexes of $\mathbf{k}$-modules with $\mathbf{k}$-linear differential raising degree by 1. The symmetric monoidal structure is given by the graded tensor product over $\mathbf{k}$:

$$\begin{aligned} (V^* \otimes W^*)^n &:= \sum_{i+j=n} V^i \otimes W^j, && (3.1)\\ d(v \otimes w) &:= d(v) \otimes w + (-1)^v v \otimes d(w), && (3.2)\end{aligned}$$

and the symmetry is given by

$$\sigma(v \otimes w) := (-1)^{vw} w \otimes v, \tag{3.3}$$

where we use the notation $(-1)^v := (-1)^{deg(v)}$, $(-1)^{vw} := (-1)^{deg(v)deg(w)}$, etc. There is a potential ambiguity in the symbol $(-1)^{vw}$ since the exponent could be either $deg(v)deg(w)$ or $deg(vw)$. In our convention it will always have the meaning $deg(v)deg(w)$.

The dual complex is defined by

$$\begin{aligned}(V^{\#})^i &:= Hom_k(V^{-i}, \mathbf{k}), &(3.4)\\ d^{\#}(\alpha) &:= (-1)^{\alpha}\alpha \circ d. &(3.5)\end{aligned}$$

The objects of category dg-$\mathtt{Col}_K$ are K-collections A such that for each n the component $A(n)$ is in dg-$\mathtt{Mod}_{\mathbf{k}}$ and the differential $d(n)^i : A(n)^i \to A(n)^{i+1}$ is left K-, right $K[n,n]$-linear. The collection A is bigraded by "arity" (or "collection degree") and by "cochain degree." The morphisms of dg-$\mathtt{Col}_K$ are morphisms $\varphi : A \to B$ of $\mathtt{Col}_K$ such that for each n, $\varphi(n) : A(n) \to B(n)$ is a morphism of dg-$\mathtt{Mod}_{\mathbf{k}}$.

The monoidal structure on dg-$\mathtt{Col}_K$ is given by the natural extension of the monoidal structure on $\mathtt{Col}_K$ defined in (1.44):

$$(A\Box_K B)(n) := \bigoplus_{j=1}^{n} A(j) \otimes_{K[j,j]} B[j,n].$$

Definition 3.1.1. *A dg-K-operad is a monoid in dg-$\mathtt{Col}_K$ relative to the $\Box_K$-monoidal structure, that is, a K-operad such that each arity-n component is a dg-$\mathbf{k}$-module and the structure maps are dg-$\mathtt{Mod}_{\mathbf{k}}$ morphisms.*

For $A \in dg$-$\mathtt{Col}_K$ the left K-, right $K[n,n]$-module on $A(n)$ defines by duality a right K-, left $K[n,n]$-module structure on $A^{\#}(n) := Hom_{\mathbf{k}}(A(n), \mathbf{k})$. If we define the algebra K^{op} with the opposite multiplication, $\mu_{\mathrm{op}}(a,b) := \mu(b,a)$ where μ is the original multiplication on K, then $A^{\#}(n)$ has a left K^{op}-, right $K^{\mathrm{op}}[n,n]$-module structure, making $A^{\#}$ a dg-K^{op}-collection with the dg-$\mathtt{Mod}_{\mathbf{k}}$ structure defined by (3.4). The K^{op}-collection structure on $A^{\#}$ allows us to use Definition 1.4.3 to define $A^{\#}_{K^{\mathrm{op}}}(T)$ for any tree T.

In explaining the cobar complex introduced by Ginzburg and Kapranov [7] it will be convenient to use the right K-module structures on $A(n)$ and right K^{op}-module structures on $A^{\#}(n)$ defined by the n imbedded subalgebras

$$\begin{aligned}K_i &:= 1 \otimes \cdots \otimes 1 \otimes K \otimes 1 \otimes \cdots \otimes 1 \hookrightarrow K[n,n],\\ K_i^{\mathrm{op}} &:= 1 \otimes \cdots \otimes 1 \otimes K_i^{\mathrm{op}} \otimes 1 \otimes \cdots \otimes 1 \hookrightarrow K_i^{\mathrm{op}}[n,n],\end{aligned}$$

where the factors of K and K^{op}, respectively, appear in position i. Let $A^{\#}(n)_i$ denote $A^{\#}(n)$ with the right K^{op}-module structure given by the natural isomorphim $K^{\mathrm{op}} \cong K_i^{\mathrm{op}}$, and denote the right action by

$$\alpha \cdot_i u := \alpha \cdot (1 \otimes \cdots \otimes 1 \otimes u \otimes 1 \otimes \cdots \otimes 1).$$

The differential on the cobar complex is defined in part by dualizing the $\circ_i$-operations. In order to make sure that all dual operations are well-defined, we consider only operads A such that $A(n)$ is a free $\mathbf{k}$-module of finite rank for all n.

Proposition 3.1.2. *If A is a K-pseudo-operad, the $\circ_i$ operations define a family of dual operations*

$$\Delta_i^{n,m} : A^{\#}(m+n-1) \longrightarrow A^{\#}(n)_i \otimes_{K^{\mathrm{op}}} A^{\#}(m), \tag{3.6}$$

$$\Delta_i^{n,m}(\lambda) := \sum \lambda_{(i,1)} \otimes \lambda_{(i,2)}, \text{ where} \tag{3.7}$$

$$\lambda(\alpha \circ_i \beta) := \sum (-1)^{\alpha\lambda_{(i,2)}} \lambda_{(i,1)}(\alpha)\lambda_{(i,2)}(\beta). \tag{3.8}$$

Proof. For any $\alpha \in A(n)$ and $\lambda \in A^{\#}(m+n-1)$ we first use the $\circ_i$ operations to define $\Delta_i^{n,m}(\lambda, \alpha) \in A^{\#}(m)$ by the equation

$$\Delta_i^{n,m}(\lambda,\alpha)(\beta) := \lambda(\alpha \circ_i \beta).$$

For a K-pseudo-operad, the K-equivariance of $\circ_i$ implies that for $u \in K$

$$\lambda((\alpha \cdot_i u) \circ_i \beta) = \lambda(\alpha \circ_i (u \cdot \beta)),$$

so we have

$$\Delta_i^{n,m}(\lambda, \alpha \cdot_i u)(\beta) = \Delta_i^{n,m}(\lambda,\alpha)(u \cdot \beta) = (\Delta_i^{n,m}(\lambda,\alpha) \cdot u)(\beta),$$

defining a morphism of right K-modules

$$\Delta_i^{n,m}(\lambda, -) : A(n)_i \longrightarrow A^{\#}(m).$$

Then

$$\Delta_i^{n,m}(\lambda, -) \in Hom_{\mathrm{right}-K}(A(n)_i, A^{\#}(m)) \cong A^{\#}(m) \otimes_K A^{\#}(n)_i \cong A^{\#}(n)_i \otimes_{K^{\mathrm{op}}} A^{\#}(m),$$

where the right K-module structure on $A(n)_i$ defines a left K-module (or a right K^{op}-module) on $A^{\#}(n)_i$. In the first isomorphism $A^{\#}(m)$ is considered as a right K-module and in the second as a left K^{op}-module. Equations (3.6), (3.7) express the fact that $\Delta_i^{n,m}(\lambda)$ is defined as the image of $\Delta_i^{n,m}(\lambda,-)$ in $A^{\#}(n)_i \otimes_{K^{\mathrm{op}}} A^{\#}(m)$ and equation (3.8) follows from the definition of $\Delta_i^{n,m}(\lambda,-)$. ■

Definition 3.1.3. *For $A \in \Psi\mathtt{Op}_K$ there are morphisms*

$$\Delta_{i,r} : A^{\#}(r) \longrightarrow \bigoplus_{\substack{m+n=r+1\\ i\le n}} A(n)_i^{\#} \otimes_{K^{\mathrm{op}}} A(m) \text{ defined by } \Delta_{i,r} := \sum_{m+n=r+1} \Delta_i^{m,n}.$$

Remark 3.1.4. For a $A \in dg$-$\Psi\mathtt{Op}_K$, the $\circ_i$ compositions are dg morphisms and so, by duality, are the $\Delta_i := \sum_r \Delta_{i,r}$. These operations define the structure of "pseudo-cooperad" dual to K-pseudo-operad, but we leave the precise formulation of the axioms to the enthusiastic reader.

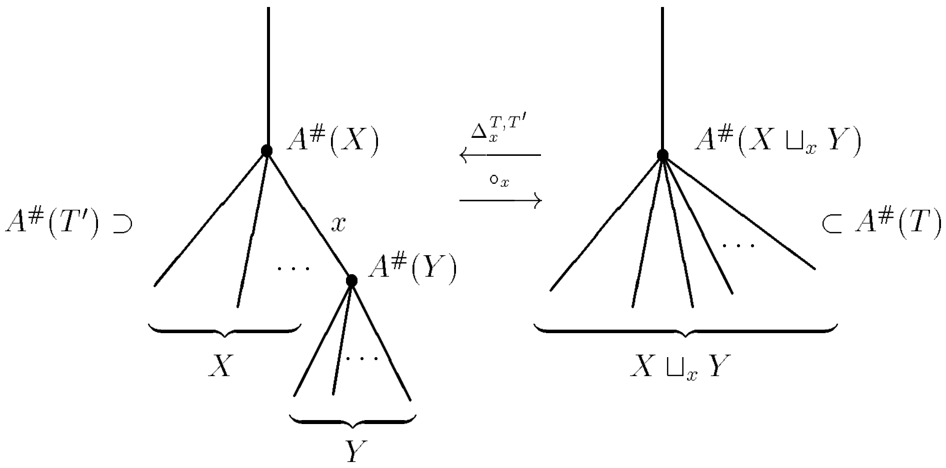


Figure 3.1: The operation $\Delta_x^{T,T'}$ dualizing $\circ_x$.

When the pseudo-operad structure is described in terms of $A(T)$ as in Theorem 1.4.4 the basic operations are the $\circ_x$-operations, one for each tree-morphism $\pi_x : T' \to T'/x = T$ such as that appearing in Figure 3.1. Define

$$X \sqcup_x Y := (X \sqcup Y) - \{x\}, \tag{3.9}$$

that is, the set resulting from replacing the element x of X with the set Y. The pseudo-operad composition $\circ_x$ is a morphism

$$\circ_x : A(X) \otimes A(Y) \longrightarrow A(X \sqcup_x Y).$$

Dualizing this operation in the same way the morphism $\Delta_i^{n,m}$ was defined in Proposition 3.1.2 we have a morphism $\Delta_x^{X,Y} : A^{\#}(X \sqcup_x Y) \to A^{\#}(X) \otimes A^{\#}(Y)$ for arbitrary sets X, Y. If we make the identifications

$$X \sqcup_x (Y \sqcup_y Z) = (X \sqcup_x Y) \sqcup_y Z \text{ and } (X \sqcup_{x_1} Y) \sqcup_{x_2} Z = (X \sqcup_{x_2} Z) \sqcup_{x_1} Y,$$

where $x, x_1, x_2 \in X$ and $y \in Y$, then the associativity axioms in equations (1.23) imply dual axioms, deleting the superscripts:

$$\begin{aligned} (id \otimes \Delta_y)\Delta_x &= (\Delta_x \otimes id)\Delta_y : A^{\#}(X \sqcup_x (Y \sqcup_y Z)) \to A^{\#}(X) \otimes A^{\#}(Y) \otimes A^{\#}(Z) \\ (\Delta_{x_1} \otimes id)\Delta_{x_2} &= (id \otimes \tau)(\Delta_{x_2} \otimes id)\Delta_{x_1} : A^{\#}((X \sqcup_{x_1} Y) \sqcup_{x_2} Z) \to A^{\#}(X) \otimes A^{\#}(Y) \otimes A^{\#}(Z). \end{aligned} \tag{3.10}$$

Given a pair of trees T, T' with $T'/x = T$, let x be an edge in T' which has a terminal vertex (the one closest to the root) with incident edges labeled by X and initial vertex with incident edges labeled by Y, i.e. the situation as in Figure 3.1. Applying $\Delta_x^{X,Y}$ at vertex $\pi(x)$ in the tensor product $A^{\#}(T)$ (where $\pi(x)$ is the vertex which is created by collapsing the edge x) defines an operation:

$$\Delta_x^{T,T'} : A^{\#}(T) \to A^{\#}(T').$$

We have the following result on extending the dg structure on $A(n)$ or $A^{\#}(n)$.

Proposition 3.1.5. *Let K be a* $\mathbf{k}$*-algebra and* $A \in dg\text{-}\mathtt{Col}_K$*, then for any tree T the dg structure on A (or $A^{\#}$) extends to $A_K(T)$ (or $A^{\#}_{K^{\mathrm{op}}}(T)$).*

Proof. We prove the proposition for $A_K(T)$. The only subtlety here is that $A_K(T)$ involves the unordered tensor product of $A(In(v))$ over the vertices of T (a colimit over the different simple orderings, see Definition 1.4.2). The differential commutes with the symmetry,

$$\begin{aligned} d\sigma(x \otimes y) &= d((-1)^{xy} y \otimes x) = (-1)^{xy} dy \otimes x + (-1)^{xy+y} y \otimes dx \\ &= \sigma((-1)^{x} x \otimes dy \,+\, dx \otimes y) = \sigma d(x \otimes y), \end{aligned}$$

therefore, it commutes with the symmetric group action permuting factors in the ordered tensor products and hence defines a differential on the colimit $A_K(T)$. ∎

Henceforth we will delete the subscripts despite the ambiguity arising from the need to deal with both K collections and K^{op} collections. When the appropriate subscript is not clear from the context, it will be written explicitly. The remainder of this section is based on [7, Section 3.2].

For any tree T, let $edg(T)$ be the set of internal edges of T and $Edg(T)$ the set of all edges including leaves. Denote the number of internal edges of T by $|T| = \#(edg(T))$.

Recall that for any $A \in dg\text{-}\mathtt{Mod}_{\mathbf{k}}$, the suspension is defined by

$$(\uparrow A)^i := A^{i-1}.$$

Definition 3.1.6. *For any finite set I let $\uparrow \mathbf{k}^I$ be the free* $\mathbf{k}$*-module with basis I considered as a dg-*$\mathbf{k}$*-module concentrated in degree 1 and define the one-dimensional dg-k-module concentrated in degree $|I|$:*

$$\det(k^I) := \wedge^{|I|}(\uparrow \mathbf{k}^I).$$

Define

$$\det(T) :=\uparrow \det(\mathbf{k}^{edg(T)}).$$

Thus $\det(T)$ is concentrated in degree $|T|+1$. An ordering $(e_1, e_2, \ldots, e_{|T|})$ of the edges of T determines a basis element $e_1 \wedge e_2 \wedge \cdots \wedge e_{|T|} \in \det(T)$.

Definition 3.1.7. *The* cobar bicomplex *of an operad A in arity n (operad-degree n) is a double complex*

$$\mathbf{C}(A)(n)^{*,*} = \bigoplus_{i,j} \mathbf{C}(A)(n)^{i,j},$$

where

$$\mathbf{C}(A)(n)^{i,j} = \bigoplus_{\{T \in \mathtt{Tree}_n,\, |T|+1=i\}} A^{\#}(T)^j \otimes \det(T),$$

where $\mathtt{Tree}_n$ *is the set of trees with leaves labeled by the set* $\{1,\ldots,n\}$. *The two differentials are* $\delta := \sum \delta^i$ *and* $\delta_A := \sum \delta_A^j$, *where*

$$\delta^i : \mathbf{C}(A)^{i,*} \longrightarrow \mathbf{C}(A)(n)^{i+1,*}$$

and

$$\delta_A^j : \mathbf{C}(A)^{*,j} \longrightarrow \mathbf{C}(A)(n)^{*,j+1}.$$

The differential

$$\delta^i : \bigoplus_{|T|=i-1} A^{\#}(T) \otimes \det(T) \longrightarrow \bigoplus_{|T'|=i} A^{\#}(T') \otimes \det(T')$$

has matrix components

$$\begin{aligned} \delta_{T,T'} &:= \Delta_e^{T,T'} \otimes e \wedge (-) : A^{\#}(T) \otimes \det(T) \to A^{\#}(T') \otimes e \wedge \det(T) \cong A^{\#}(T') \otimes \det(T'), \\ &\quad\ \text{if } T = T'/e \\ \delta_{T,T'} &:= 0 \quad \text{if } T \neq T'/e. \end{aligned}$$

We call δ *the* tree differential *and* j *the* tree degree.

The second differential δ_A *is called the* internal differential *where*

$$\delta_A^j = d_{A\#}^j \otimes id : A^{\#}(T)^j \otimes \det(T) \longrightarrow A^{\#}(T)^{j+1} \otimes \det(T),$$

and $d_{A\#}^j$ *is defined in Proposition 3.1.5. The* cobar complex *is the total complex of the cobar bicomplex with differential* $d = \delta + (-1)^{|T|}\delta_A$.

The δ-complex is

$$A^{\#}(n)^* \xrightarrow{\delta} \bigoplus\nolimits_{|T|=1} A^{\#}(T)^* \otimes \det(T) \xrightarrow{\delta} \cdots \xrightarrow{\delta} \bigoplus\nolimits_{|T|=n-2} A^{\#}(T)^* \otimes \det(T), \tag{3.11}$$

where the summand corresponding to the tree T has tree degree $|T|+1$ and the condition $T \in \mathtt{Tree}_n$ is implicit in all the direct sums. The condition $\delta^2 = 0$ follows immediately from the coassociativity identities (3.10).

At the end of this section we give a more explicit description of the structure of $\mathbf{C}(A)(n)$ as a right $\mathbf{k}[\Sigma_n]$-module.

Theorem 3.1.8. *There is an isomorphism of dg-*K^{op}*-modules between the cobar complex graded by total degree with differential* δ_A *and the free operad on the collection* $\uparrow A^{\#}$*:*

$$(\mathbf{C}(A)(n), \delta_A) \cong \Gamma(\uparrow A^{\#})(n) \tag{3.12}$$

with the induced differential. Therefore, the cobar complex defines a contravariant functor $C : dg\text{-}\mathtt{Op}_K \to dg\text{-}\mathtt{Op}_{K^{\mathrm{op}}}$. *Moreover this functor preserves quasi-isomorphims.*

Proof. It is clear that if we ignore the factors $\det(T)$ and degrees, then $(\mathbf{C}(A)(n), \delta_A)$ is isomorphic to $(\Gamma(A^{\#})(n), d_{A^{\#}})$. The following lemma ([7, Lemma 3.2.9]) describes what happens when we include the determinants.

Lemma 3.1.9. *For any finite set I of m elements and $W_i \in dg\text{-}\mathtt{Mod}_{\mathbf{k}}$, $i \in I$, there is a canonical isomorphism*

$$\varphi : \bigotimes_{i\in I}(\uparrow W_i) \to \bigotimes_{i\in I} W_i \otimes \det(\mathbf{k}^I).$$

For $A \in dg\text{-}\mathtt{Col}_K$ and $T \in \mathtt{Tree}_m$ there is a canonical isomorphism

$$(\uparrow A)(T) \cong A(T) \otimes \det(T).$$

Proof. If we choose an ordering $\{i_1, \ldots, i_m\}$ of I and represent $\bigotimes_{i\in I} W_i$ as $W_{i_1} \otimes \cdots \otimes W_{i_m}$, then we can define $\varphi : \bigotimes_{1\le j\le m} \uparrow W_{i_j} \to \bigotimes_{1\le j\le m} W_{i_j} \otimes \det(\mathbf{k}^I)$ by

$$\varphi(\uparrow w_1 \otimes \cdots \otimes \uparrow w_m) = (-1)^{\sum_{1\le i\le m} i w_i}(w_1 \otimes \cdots \otimes w_m) \otimes (i_1 \wedge \cdots \wedge i_m), \tag{3.13}$$

where $\uparrow w_j \in \uparrow W_{i_j}$ and $w_j \in W_{i_j}$ are corresponding elements. By the definition of $\det(\mathbf{k}^I)$, φ preserves degree. The following equations show that φ commutes with d:

$$\begin{aligned}
&\varphi(d(\uparrow w_1 \otimes \cdots \otimes \uparrow w_m)) \\
&\quad = \sum_{1\le j\le m} (-1)^{\left(\sum_{i<j}(w_i+1)\right)} \varphi(\uparrow w_1 \otimes \cdots \otimes \uparrow dw_j \otimes \cdots \otimes \uparrow w_m) \\
&\quad = \sum_{1\le j\le m} (-1)^{(\sum_{i\ne j} i w_i + j(w_j+1) + \sum_{i<j} w_i + j)} (w_1 \otimes \cdots \otimes dw_j \otimes \cdots \otimes w_m) \otimes (i_1 \wedge \cdots \wedge i_m) \\
&\quad = \sum_{1\le j\le m} (-1)^{(\sum i w_i + \sum_{i<j} w_i)} (w_1 \otimes \cdots \otimes dw_j \otimes \cdots \otimes w_m) \otimes (i_1 \wedge \cdots \wedge i_m) \\
&\quad = d\varphi(\uparrow w_1 \otimes \cdots \otimes \uparrow w_m)).
\end{aligned}$$

Next we show that φ commutes with permutations and hence is well-defined independent of the choice of ordering of I. It is enough to consider σ_j transposing factors w_j, w_{j+1},

$$\begin{aligned}
\varphi(\sigma_j(\uparrow w_1 \otimes \cdots \otimes \uparrow w_m))
&= (-1)^{(w_j+1)(w_{j+1}+1)} \varphi(\uparrow w_1 \otimes \cdots \otimes \uparrow w_{j+1} \otimes \uparrow w_j \otimes \cdots \otimes \uparrow w_m) \\
&= (-1)^{\sum i w_i - w_{j+1} + w_j + (w_j w_{j+1} + w_j + w_{j+1} + 1)} \\
&\qquad \times (w_1 \otimes \cdots \otimes w_{j+1} \otimes w_j \cdots \otimes w_m) \otimes (i_1 \wedge \cdots \wedge i_{j+1} \wedge i_j \cdots \wedge i_m) \\
&= (-1)^{(\sum i w_i + w_j w_{j+1} + 1)} (w_1 \otimes \cdots \otimes w_{j+1} \otimes w_j \cdots \otimes w_m) \\
&\qquad\qquad \otimes (i_1 \wedge \cdots \wedge i_{j+1} \wedge i_j \cdots \wedge i_m) \\
&= \sigma_j \left((-1)^{\sum i w_i}(w_1 \otimes \cdots \otimes w_m)\right) \otimes (i_1 \wedge \cdots \wedge i_j \wedge i_{j+1} \cdots \wedge i_m) \\
&= (\sigma_j \otimes id)\varphi(\uparrow w_1 \otimes \cdots \otimes \uparrow w_m)).
\end{aligned}$$

Therefore φ commutes with all permutations and generates a morphism of the colimit defining the unordered tensor product.

To prove the second statement in the Lemma, note that there is canonical identification of $\bigwedge^{|T|+1}(\mathbf{k}^{V(T)})$ and $\bigwedge^{|T|}(\mathbf{k}^{edg(T)})$ coming from the splitting

$$\mathbf{k}^{V(T)} \cong \mathbf{k}^{edg(T)} \oplus \mathbf{k}\iota_{\mathrm{root}},$$

related to the decomposition $V(T) = edg(T) \sqcup \{\mathrm{root}\}$, where $\iota_{\mathrm{root}} \in \mathbf{k}^{V(T)}$ has value one on the root vertex and zero on all others. Then there is a dg-isomorphism

$$(\uparrow A)(T) \cong A(T) \otimes \det(\mathbf{k}^{V(T)}) \cong A(T) \otimes \uparrow \det(\mathbf{k}^{edg(T)}) = A(T) \otimes \det(T).$$

■

To complete the proof of Theorem 3.1.8, we need only note that the $\circ_i$-operations are clearly morphisms in dg-$\mathtt{Mod}_{\mathbf{k}}$ so $\mathbf{C}(A) \in dg$-$\mathtt{Op}_K$. The functor $\mathbf{C}$ preserves quasi-isomorphisms because the same is true of the functor $A \to A^{\#}(T)$ for any tree T and δ is a cochain map relative to the internal differentials. ■

The complex $\mathbf{D}(A)$ which we define below is a twisted form of the cobar complex with cochain degree shifted by $-n+1$ so that the maximal cochain degree is normalized to be 0. Both complexes $\mathbf{C}(A)$ and $\mathbf{D}(A)$ are necessary for the study of quadratic operads and the Koszul property which is the topic of the next section. For some purposes it will be convenient to use $\mathbf{C}(A)$ rather than $\mathbf{D}(A)$. The construction of $\mathbf{D}(A)$ uses the notions of operadic suspension and desuspension, which will be important in a number of other contexts as well.

Definition 3.1.10. *For any dg-operad A, the* operadic suspension $\mathfrak{s}A$ *is the dg-operad whose arity n component is given by*

$$\mathfrak{s}A(n) := \uparrow^{n-1} A(n) \otimes \mathrm{sgn}_n.$$

Similarly we introduce the operadic desuspension $\mathfrak{s}^{-1}A$*:*

$$\mathfrak{s}^{-1}A(n) := \downarrow^{n-1} A(n) \otimes \mathrm{sgn}_n.$$

Remark 3.1.11. Note that

$$\mathfrak{s}^{-1}End_V \cong End_{\uparrow V},$$

since

$$\begin{aligned} Hom\left((\uparrow V)^{\otimes n}, \uparrow V\right)^i &= \bigoplus_j Hom\left(((\uparrow V)^{\otimes n})^j, \uparrow V^{i+j}\right) = \bigoplus_j Hom((V^{\otimes n})^{j-n}, V^{i+j-1}) \\ &= Hom(V^{\otimes n}, V)^{i+n-1} \subset (\mathfrak{s}^{-1}End_V(n))^i. \end{aligned}$$

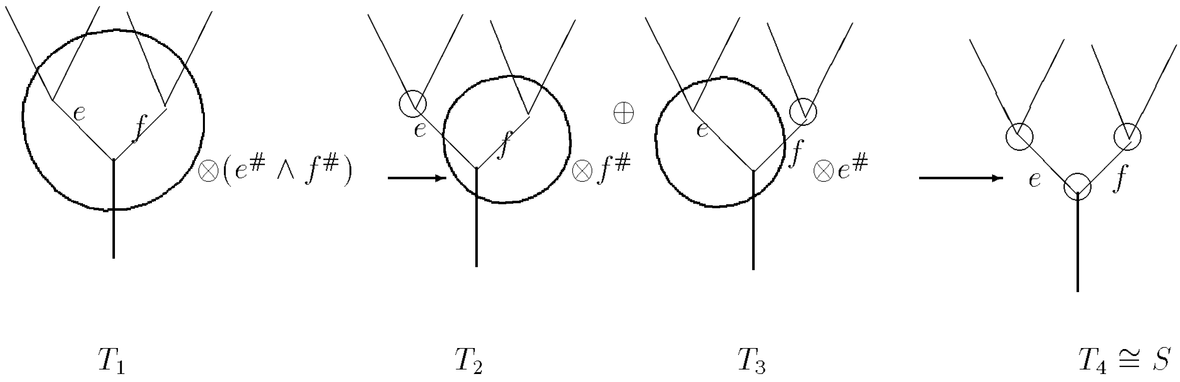


Figure 3.2: The complex $\mathbf{D}(\mathbf{D}(A))$ with differential δ_1 in the case of a binary tree S with three vertices. The trees T_i have the encircled subtrees collapsed to vertices, so T_1 is a corolla and T_4 is isomorphic to S.

Definition 3.1.12. *The* dual *dg*-operad $\mathbf{D}(A)$ *is the operadic desuspension of* $\mathbf{C}(A)$*:*

$$\mathbf{D}(A) := \mathfrak{s}^{-1}\mathbf{C}(A).$$

Ginzburg and Kapranov [7] construct $\mathbf{D}(A)$ using what they call the determinant operad.

Definition 3.1.13. *The* determinant operad $\bigwedge$ *is the desuspension of the commutative operad,*

$$\bigwedge := \mathfrak{s}^{-1}Com.$$

In other words, the arity n component $\bigwedge(n)$ is a 1-dimensional complex concentrated in internal degree $1-n$ carrying the signum representation of Σ_n.

Remark 3.1.14. The tensor product of operads $\mathbf{C}(A)\otimes Com$ defined by $(\mathbf{C}(A)\otimes Com)(n) := \mathbf{C}(A)(n) \otimes Com(n)$ is naturally isomorphic to $\mathbf{C}(A)$. Therefore, since tensoring with Com followed by $\mathfrak{s}^{-1}$ is equivalent to tensoring with $\mathfrak{s}^{-1}Com$,

$$\mathbf{D}(A) = \mathfrak{s}^{-1}\mathbf{C}(A) \cong \mathfrak{s}^{-1}(\mathbf{C}(A) \otimes Com) \cong \mathbf{C}(A) \otimes \mathfrak{s}^{-1}Com \cong \mathbf{C}(A) \otimes \bigwedge.$$

The last term is the definition of $\mathbf{D}(A)$ in [7].

Definition 3.1.15. *Let $E(T) = \#(Edg(T))$ and $L(T) = \#(leaves(T))$ and define*

$$\mathrm{Det}(T)^i = \begin{cases} \wedge^{E(T)}(\mathbf{k}^{Edg(T)}) & \text{for } i = |T| - L(T) + 2, \\ 0 & \text{for } i \neq |T| - L(T) + 2. \end{cases}$$

With this definition, the complex $\mathbf{D}(A)$ relative to δ is

$$A^*(n) \xrightarrow{\delta} \bigoplus\nolimits_{|T|=1} A^*(T) \otimes \mathrm{Det}(T) \xrightarrow{\delta} \cdots \xrightarrow{\delta} \bigoplus\nolimits_{|T|=n-2} A^*(T) \otimes \mathrm{Det}(T), \tag{3.14}$$

with tree-degree of the term corresponding to a tree T equal to $|T| + 2 - n$, so the last term on the right has tree-degree zero and the complex is negatively graded. The Σ_n-action is described in more detail at the end of this section.

Theorem 3.1.16. *If A is a dg-K-operad such that each $A(n)$ is a finite dimensional free $\mathbf{k}$-module, then there is a canonical quasi-isomorphism $\mathbf{D}(\mathbf{D}(A)) \longrightarrow A$.*

Proof. The complex $\mathbf{D}(\mathbf{D}(A))$ is tri-graded, with multi-index (i, j, k). The index i is the tree-degree coming from the last application of the functor $\mathbf{D}$, and corresponds to a tree differential denoted by δ_1. The index j is the *negative* of the tree-degree coming from the first application of $\mathbf{D}$, negative because of the passage to $\mathbf{D}(A)^\#$ in the definition of $\mathbf{D}(\mathbf{D}(A))$. Since the corresponding partial differential δ_2 is dual to the tree differential on $\mathbf{D}(A)$ it expresses the composition law on the operad A. The last index k indicates the internal *dg*-degree in A and has corresponding partial differential d_A. Let $\mathtt{n}(v) := \#In(v)$, then by definition,

$$\begin{aligned}
\mathbf{D}(\mathbf{D}(A))^{i,j,k}(n) &= \bigoplus_{\substack{T\in\mathtt{Tree}_n\\ |T|+2-n=i}} \mathbf{D}(A)^\#(T)^{j,k} \otimes \mathrm{Det}(T)\\
&\cong \bigoplus_{\substack{T\in\mathtt{Tree}_n\\ |T|+2-n=i}} \left(\bigotimes_{v\in V(T)} \mathbf{D}(A)^\#(In(v))\right)^{j,k} \otimes \mathrm{Det}(T)\\
&\cong \bigoplus_{\substack{T\in\mathtt{Tree}_n\\ |T|+2-n=i}} \left(\bigotimes_{\substack{v\in V(T)\\ \sum j_v=j,\sum k_v=k}} \left(\bigoplus_{\substack{S_v\in\mathtt{Tree}_{In(v)}\\ |S_v|+2-\mathtt{n}(v))=-j_v}} (A(S_v)^{k_v})^\# \otimes \mathrm{Det}(S_v)\right)^\# \right) \otimes \mathrm{Det}(T)\\
&\cong \bigoplus_{\substack{T\in\mathtt{Tree}_n\\ |T|+2-n=i}} \left(\bigotimes_{\substack{v\in V(T)\\ \sum j_v=j,\sum k_v=k}} \left(\bigoplus_{\substack{S_v\in\mathtt{Tree}_{In(v)}\\ |S_v|+2-\mathtt{n}(v))=-j_v}} A(S_v)^{k_v} \otimes \mathrm{Det}(S_v)^\# \right) \right) \otimes \mathrm{Det}(T).
\end{aligned}$$

These equivalences are explained as follows. Since $A(n)^k$ is finite dimensional for all n, k, and $^\#$ indicates the graded dual, we have $((A(S_v)^k)^\#)^\# \cong A(S_v)^k$. The index i is non-positive and equals $|T| + 2 - n$, reflecting the fact that $\mathfrak{s}^{-1}$ lowers cobar degree by $n - 1$. Thus $i = 0$ corresponds to the binary trees in $\mathtt{Tree}_n$ ($n - 1$ vertices). The *dg*-K-module $\mathbf{D}(A)(T)$ consists of tensor products of elements of $\mathbf{D}(A)$, one for each vertex v of T, and we use j_v to denote cochain degree of the element associated to vertex v. The elements of $\mathbf{D}(A)$ themselves correspond to labeled trees, so there is a tree $S_v \in \mathtt{Tree}_{In(v)}$ assigned to each vertex of T. The relation between the number of edges of S_v and the degree j_v is $j_v = -(|S_v| + 2 - \mathtt{n}(v))$.

Starting from the trees $\{S_v \in \mathtt{Tree}_{In(v)} | v \in V(T)\}$, we construct a "big tree" S by grafting the tree S_v into T at the vertex v. Then there is a morphism of trees $\pi : S \to T$ such that $S_v = \pi^{-1}(v)$ and π contracts each subtree S_v onto the corresponding vertex v in T. The vertices of S are partitioned into the sets $V(S_v)$ indexed by the vertices of T, thus the vertices of T can be identified with subsets which constitute the partition. As an example, consider Figure 3.2 where the same big tree S (a binary tree with three vertices) is associated to four trees. Each of the trees T_i corresponds to a partition of the vertices of S into subsets consisting of the vertices of a subtree.

Coming back to the equation above, we can reexpress the last line:

$$
\begin{aligned}
& \bigoplus_{\substack{T \in \mathtt{Tree}_n \\ |T|+2-n=i}} \left(\bigotimes_{\substack{v \in V(T) \\ \sum j_v = j, \sum k_v = k}} \left(\bigoplus_{\substack{S_v \in \mathtt{Tree}_{In(v)} \\ |S_v|+2-\mathtt{n}(v) = -j_v}} A(S_v)^{k_v} \otimes \mathrm{Det}(S_v)^{\#} \right) \right) \otimes \mathrm{Det}(T) \\
= & \bigoplus_{\substack{\pi : S \to T \\ |T|+2-n=i,\ |S|+2-n=-j}} \left(\bigotimes_{\substack{v \in V(T),\ S_v = \pi^{-1}(v) \\ w \in V(S_v), \sum k_{v,w} = k}} A(In(w))^{k_{v,w}} \otimes \mathrm{Det}(S_v)^{\#} \right) \otimes \mathrm{Det}(T) \\
\cong & \bigoplus_{\substack{\pi : S \to T \\ |T|+2-n=i,\ |S|+2-n=-j}} A(S)^k \otimes \Big(\bigotimes_{v \in V(T), S_v = \pi^{-1}(v)} \mathrm{Det}(S_v)^{\#} \Big) \otimes \mathrm{Det}(T) \\
\cong & \bigoplus_{\substack{\pi : S \to T \\ |T|+2-n=i,\ |S|+2-n=-j}} A(S)^k \otimes \Big(\bigotimes_{v \in V(T), S_v = \pi^{-1}(v)} \det(S_v)^{\#} \Big).
\end{aligned}
$$

The substitution of $\bigotimes \det(S_v)^{\#}$ in the last line is a consequence of the following isomorphisms (compare the definition of a "coefficient system" for modular operads in Section 3.1):

$$
\mathrm{Det}(T) \otimes \left(\bigotimes_{v \in V(T)} \det(S_v) \right) \cong \mathrm{Det}(S) \cong \bigotimes_{v \in V(T)} \mathrm{Det}(S_v).
$$

By duality we get

$$
\mathrm{Det}(T) \otimes \left(\bigotimes_{v \in V(T)} \mathrm{Det}(S_v)^{\#} \right) \cong \bigotimes_{v \in V(T)} \det(S_v)^{\#}.
$$

The new indexing $|S| + 2 - n = -j$ in the last three lines is explained by the following sequence of identities:

$$
-j = \sum_{v \in V(T)} -j_v \quad = \quad \sum_{v \in V(T)} (|S_v| + 2 - \mathtt{n}(v))
$$

$$
\begin{aligned}
&= \sum_{v \in V(T)} (|V(S_v)| + 1 - \mathtt{n}(v)) \\
&= |V(S)| + |V(T)| - \sum_{v \in V(T)} \mathtt{n}(v) \\
&= |V(S)| + |V(T)| - (|T| + n) = |V(S)| + 1 - n = |S| + 2 - n.
\end{aligned}
$$

In order to describe the differential

$$\delta_1 : \mathbf{D}(\mathbf{D}(A))^{(i,j,k)} \longrightarrow \mathbf{D}(\mathbf{D}(A))^{(i+1,j,k)}$$

it is useful to represent π in terms of the edges which it contracts, $\pi : S \to S/\{e_1, e_2, \ldots, e_k\} = T$. Given a tree $T' = S/\{e_1, \ldots, e_{j-1}, e_{j+1}, \ldots, e_k\}$, such that $T = T'/e_j$ and the edge e_j is contracted to the vertex v in T, the component $\delta_1^{T,T'}$ (dual to $\mathbf{D}(A)(T') \to \mathbf{D}(A)(T)$) consists of replacing the vertex v of T by the edge e_j of S_v and expressing S_v as the grafting of two trees along the edge e_j. The operation does not change the big tree S and does not affect the labels from the operad A, it merely repartitions the vertices of S, see Figure 3.2. From this description it becomes clear that the entire δ_1 complex can be decomposed into a direct sum over subcomplexes $\mathbf{D}(\mathbf{D}(A))_S$ indexed by "big trees" S,

$$\mathbf{D}(\mathbf{D}(A))_S^{(i,n-2-|S|,k)} := \bigoplus_{\substack{\pi:S\to T \\ |T|+2-n=i}} A(S)^k \otimes \bigotimes_{v\in V(T)} \det(S_v)^{\#}.$$

The trees T which appear in this subcomplex arise from all possible images of S under a tree-morphism. Figure 3.2 shows a typical example. For any subset of internal edges $J \subset edg(S)$, there is a tree $T_J = S/J$, with $|T_J| = |S| - |J| = i + n - 2$. In terms of the subsets J we have

$$
\begin{aligned}
\mathbf{D}(\mathbf{D}(A))_S^{(i,n-2-|S|,k)} &= \bigoplus_{\substack{J\subset edg(S) \\ |S|-|J|=i+n-2}} A(S)^k \otimes \bigotimes_{v\in V(T_J)} \det(S_v)^{\#} \\
&\cong \bigoplus_{\substack{J=\{e_{i_1},\cdots,e_{i_{|J|}}\}\subset edg(S) \\ |J|=|S|-i-n+2}} A(S)^k \otimes \langle e_{i_1}^{\#} \wedge \cdots \wedge e_{i_{|J|}}^{\#} \rangle.
\end{aligned}
$$

The term $e_{i_1}^{\#} \wedge \cdots \wedge e_{i_{|J|}}^{\#}$ is the wedge product of all the internal edges in all the subtrees S_v and δ_1 decreases $|J|$ by one.

For the corolla S with n leaves and no internal edges, $|T| = |S| = 0$, $i = 2 - n$ and $j = n - 2$. Since there are no internal edges in S, both δ_1 and δ_2 (coming from dualizing the first application of $\mathbf{D}$) are zero. Thus for each $n > 0$,

$$\Big(\mathbf{D}(\mathbf{D}(A))(n)^{(2-n,n-2,*)}, \delta\Big) \cong (A(n)^*, d_{A(n)}).$$

If we prove that for any tree S other than a corolla the corresponding subcomplex $(\mathbf{D}(\mathbf{D}(A))_S, \delta_1)$ is acyclic, then, since the total differential can be expressed as the differential of a double complex,

$$d_{\mathbf{D}(\mathbf{D}(A))} = \sum \delta_1^i + (-1)^i \delta_{\mathbf{D}(A)},$$

the acyclicity follows from a standard argument in the theory of double complexes (see for example [19]).

The cohomology is concentrated in the complementary subcomplex given by the direct sum over corollae S and the quasi-isomorphism referred in the statement of the theorem is just projection on that subcomplex.

To prove δ_1 acyclicity for $|S| > 0$, we will show that in this case the subcomplex $(\mathbf{D}(\mathbf{D}(A))_S, \delta_1)$ is isomorphic to the augmented chain complex of a $|S|-1$ simplex tensored with $A(S)$.

For each tree S define

$$\xi_S : \mathbf{D}(\mathbf{D}(A))_S \longrightarrow A(S) \otimes \bigwedge(\{e_i^{\#} | 1 \leq i \leq |S|\})$$

by $\xi_S := id \otimes \mu_\wedge$, where

$$\mu_\wedge : \bigotimes_{v \in V(T_J)} \det(S_v)^{\#} \longrightarrow \det(S)^{\#}$$

is the wedge product. Then ξ_S is a sum $\xi_S = \sum \xi_{S,J}$, where for $J = \{e_{i_1}, \dots, e_{i_{|J|}}\}$,

$$\xi_{S,J} : A(S) \otimes (\bigotimes_{v \in V(T_J)} \det(S_v)^{\#}) \longrightarrow A(S) \otimes \langle e_{i_1}^{\#} \wedge \cdots \wedge e_{i_j}^{\#} \rangle.$$

It is clear from the discussion above that

$$\delta_1 : A(S) \otimes \bigotimes_{v \in V(T_J)} \det(S_v)^{\#} \longrightarrow \bigoplus_{\substack{1 \leq k \leq |J| \\ J_k = J - e_{i_k}}} A(S) \otimes \bigotimes_{v \in V(T_{J_k})} \det(S_v)^{\#}$$

is a sum of terms each one given by deleting one edge from the set J and it intertwines with ξ_S as follows

$$\xi_S \circ \delta_1 = \left(\sum_{1 \leq i \leq |S|} e_i \rfloor \right) \circ \xi_S,$$

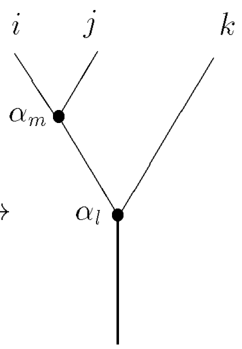
where the operation $\rfloor$ contracts e_i with $e_i^{\#}$. The chain complex $(\bigwedge\{e_i^{\#}\}, d := (\sum e_i \rfloor))$ is clearly isomorphic to the augmented simplicial chain complex of the $|S|-1$ simplex, which is acyclic. ■

3.1.1. Appendix to Section 3.1.

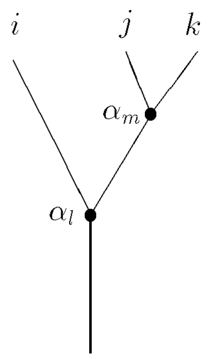
In some calculations with $\Gamma(A)(n)$ and $\mathbf{D}(A)(n)$, in particular those involving the Σ_n-action, it is convenient to replace the indexing set consisting of abstract n-trees with planar n-trees. For planar trees the orientation of the plane orders the edges at each vertex and the labels $A(In(v))$ can be replaced with the simpler object $A(\mathtt{n}(v))$, where $\mathtt{n}(v) := \# In(v)$. Moreover, the vertices of a planar tree have a natural order which we call the *planar order*. In the planar order the root vertex is first, followed by the vertices at distance one from the root (connected by one edge to the root) in order from left to right, next in order are vertices at distance two, listed from left to right, and so on. In this way all tensor products over the set of vertices have a natural order. The drawback is that the representation of elements is not unique since different planar trees may be isomorphic as abstract trees.

For example, there is only one binary (rooted) tree with three leaves, but there are two planar binary trees with three leaves. For each of the planar trees there are six labelings of the leaves given by the six permutations (i, j, k) of $(1, 2, 3)$. For a collection A with basis $\{\alpha_l\}$ in degree 2 (and assuming $A(3) = 0$) the arity 3 component $\Gamma(A)(3)$ is generated by the expressions represented by the following labeled planar trees:

$(\alpha_l \otimes_1 \alpha_m) \otimes (i, j, k) \leftrightarrow$



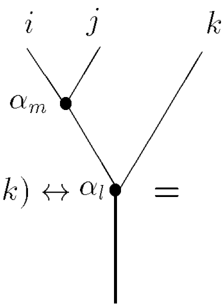
$(\alpha_l \otimes_2 \alpha_m) \otimes (i, j, k) \leftrightarrow$



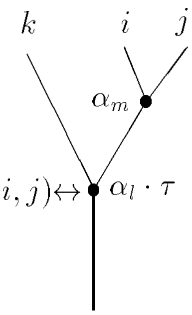
(3.15)

In order to have a basis for $\Gamma(A)(3)$ we need to take into account equivalences such as the one illustrated in the following picture.

$(\alpha_l \otimes_1 \alpha_m) \otimes (i, j, k) \leftrightarrow$



$= \quad (\alpha_l \cdot \tau \otimes_2 \alpha_m) \otimes (k, i, j) \leftrightarrow$



(3.16)

where $\alpha_l, \alpha_m \in A(2)$ and τ is the generator of Σ_2.

This reduction procedure shows that there is a basis for $\Gamma(A)(3)$ using only the set pictured on the left (or alternatively, only those pictured on the right). It is always possible to restrict to one such set, say the elements of the form $(\alpha \otimes_1 \alpha') \otimes (i,j,k)$ for α, α' in a basis for $A(2)$ and $(i,j,k) \in \Sigma_3$, therefore, unless indicated otherwise we use the notation $(\alpha \otimes \alpha') \otimes (i,j,k)$ to stand for $(\alpha \otimes \alpha') \otimes_1 (i,j,k)$ and assume the identity

$$(\alpha \otimes \alpha') \otimes (i,j,k) = (\alpha \otimes \alpha' \cdot \tau) \otimes (j,i,k).$$

There are occasions where this is not the most convenient form (c.f., the example of the associative operad given in the next section) and we will also use the notation $\otimes_1$ and $\otimes_2$.

The expressions $(\alpha_l \otimes \alpha_m) \otimes (i,j,k)$ span a representation of Σ_3 induced from the representation of Σ_2 acting on the second factor of $A(2) \otimes A(2)$. To understand the structure of the induced representation in relation to the labeling of the leaves, consider first the case $n = 3$ and the examples above. The term $(\alpha_l \otimes \alpha_m) \otimes (i,j,k)$ can be thought of as a symbol for function defined on (x_1, x_2, x_3),

$$((\alpha_l \otimes \alpha_m) \otimes (i,j,k))\,(x_1, x_2, x_3) \leftrightarrow \alpha_l(\alpha_m(x_i, x_j), x_k).$$

The right Σ_3-action is dual to the left action on the triple (x_1, x_2, x_3),

$$\begin{aligned} ((\alpha_l \otimes \alpha_m) \otimes (i,j,k))\sigma)\,(x_1,x_2,x_3) &= ((\alpha_l \otimes \alpha_m) \otimes (i,j,k))\,(\sigma(x_1,x_2,x_3)) \\ &= ((\alpha_l \otimes \alpha_m) \otimes (i,j,k))\,(x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, x_{\sigma^{-1}(3)}) \\ &= ((\alpha_l \otimes \alpha_m) \otimes (\sigma^{-1}(i), \sigma^{-1}(j), \sigma^{-1}(k)))(x_1,x_2,x_3). \end{aligned}$$

In the example of the planar binary tree with 3 leaves shown on the left side of (3.15) there is an automorphism which transposes the first two leaves. This automorphism acts on the left on the labels by $\tau(i,j,k) := (j,i,k)$. If we impose the relations

$$(\alpha_l \otimes \alpha_m) \otimes \tau(i,j,k) = (\alpha_l \otimes \alpha_m) \cdot \tau \otimes (i,j,k) = (\alpha_l \otimes \alpha_m \cdot \tau) \otimes (i,j,k) \tag{3.17}$$

then it is clear that the expressions $(\alpha_l \otimes \alpha_m) \otimes (i,j,k)$ where (i,j,k) represents an element of Σ_3 and $\tau \in \Sigma_2$ satisfy the relations for tensor product over $\mathbf{k}[\Sigma_2]$. In other words, the relations (3.17) define the induced representation

$$(A(2) \otimes A(2)) \otimes_{\mathbf{k}[\Sigma_2]} \mathbf{k}[\Sigma_3],$$

with $\mathbf{k}[\Sigma_2]$ acting on the second $A(2)$ factor.

In general case of arity n the right action of $\sigma \in \Sigma_n$ on the label $(i_1, \ldots, i_n)$ is

$$(i_1, \ldots, i_n)\sigma := (\sigma^{-1}(i_1), \ldots, \sigma^{-1}(i_n)). \tag{3.18}$$

As a matter of notation, we can represent the elements of Σ_n as $(\sigma^{-1}(1), \ldots, \sigma^{-1}(n))$, $(\sigma(1), \ldots, \sigma(n))$, or the cycle notation in terms of disjoint cycle decomposition. Since the first description seems most natural for the coefficient representing the labeling of the leaves of a planar tree, that will be the one we use. Right multiplication of σ by ρ takes the form

$$(\sigma^{-1}(1), \ldots, \sigma^{-1}(n))\rho = (\rho^{-1}\sigma^{-1}(1), \ldots, \rho^{-1}\sigma^{-1}(n)),$$

consistent with the conventions above.

There is also a left action of Σ_n on the labels given by

$$\sigma(i_1, \ldots, i_n) := (i_{\sigma^{-1}(1)}, \ldots, i_{\sigma^{-1}(n)}). \tag{3.19}$$

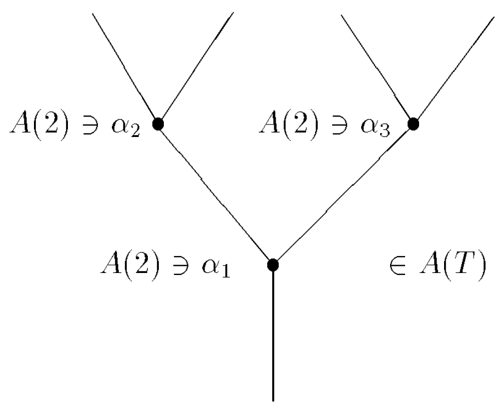
Clearly, the left and right actions commute.

The general description of $A(T)$ as an induced representation is based on the following definition.

Definition 3.1.17. *For any planar tree T with n leaves, label the leaves in sequence from left to right by 1 to n. A tree isomorphism leaving T invariant as a planar tree permutes the labels. In this way we define a monomorphism from automorphism group of the planar tree T into Σ_n and identify the tree automorphism group with its image $\Sigma(T) \subset \Sigma_n$.*

The action of tree morphisms on the labels $\{1, \ldots, n\}$ is given by restriction of the left Σ_n-action to $\Sigma(T)$.

For the binary tree pictured below $\Sigma(T)$ is the subgroup of Σ_4 of order 8 generated by the permutations (2134), (1243) and (3412) (using the notation explained above).



In this example the action of the subgroup $\Sigma(T)$ on $A(2) \otimes A(2) \otimes A(2)$ is determined by the following rules for the generators:

$$\begin{aligned}
(\alpha_1 \otimes \alpha_2 \otimes \alpha_3) \cdot (2134) &= \alpha_1 \otimes (\alpha_2 \cdot \tau) \otimes \alpha_3 \\
(\alpha_1 \otimes \alpha_2 \otimes \alpha_3) \cdot (1243) &= \alpha_1 \otimes \alpha_2 \otimes (\alpha_3 \cdot \tau) \\
(\alpha_1 \otimes \alpha_2 \otimes \alpha_3) \cdot (3412) &= (-1)^{\alpha_2\alpha_3}(\alpha_1 \cdot \tau) \otimes \alpha_3 \otimes \alpha_2
\end{aligned} \tag{3.20}$$

Given a planar tree T with vertices $(v_0, \ldots, v_{|T|})$ in planar order there is an obvious generalization of the equations (3.20) to define an action of $\Sigma(T)$ on

$$A\langle T\rangle := A(\mathtt{n}(v_0)) \otimes \cdots \otimes A(\mathtt{n}(v_{|T|})). \tag{3.21}$$

This leads to the following proposition. The proof is straightforward and is left to the reader.

Proposition 3.1.18. *The Σ_n-module given by the direct sum over all labelings can be described as an induced representation:*

$$A(T) \cong A\langle T\rangle \otimes_{\mathbf{k}[\Sigma(T)]} \mathbf{k}[\Sigma_n]. \tag{3.22}$$

We defined $A(T)$ as an unordered tensor product, that is, a colimit over the different orderings of the vertices of the abstract n-tree T. The choice of a planar n-tree representing an abstract tree is equivalent to specifying an ordering for the vertices and using ordinary tensor products. The functor $\Gamma(A)(n)$ is defined as a direct sum over the indexing set of abstract n-trees $\mathtt{Tree}_n$. By choosing a planar tree representing each abstract tree, we can index the direct sum by a set $\mathtt{Tree}_n^{\mathrm{pl}}$ consisting of planar n-trees, no two of which are isomorphic as abstract trees,

$$\Gamma(A)(n) \cong \bigoplus_{T \in \mathtt{Tree}_n^{\mathrm{pl}}} A\langle T\rangle \otimes_{\mathbf{k}[\Sigma(T)]} \mathbf{k}[\Sigma_n]. \tag{3.23}$$

This description of $\Gamma(A)(n)$ will be helpful in proving the following proposition relating the suspension functor to the free operad functor.

Proposition 3.1.19. *The suspension functor and the free operad functor commute. There is a natural transformation of functors from dg-$\mathtt{Col}_K$ to dg-$\mathtt{Op}_K$:*

$$\Phi : \mathfrak{s} \circ \Gamma \longrightarrow \Gamma \circ \mathfrak{s}.$$

The proof of the proposition reduces to the proof of the following lemma.

Lemma 3.1.20. *Let T be a planar n-tree, $A \in$dg-$\mathtt{Col}_K$, and define*

$$\mathfrak{s}(A\langle T\rangle) := \uparrow^{n-1} A\langle T\rangle \otimes \mathrm{sgn}_n.$$

Then there is an $\Sigma(T)$-equivariant isomorphism

$$\mathfrak{s}(A\langle T\rangle) \cong (\mathfrak{s}A)\langle T\rangle$$

which commutes with morphisms in dg-$\mathtt{Col}_K$.

For an abstract tree $T \in \mathtt{Tree}_n$, if we define

$$\mathfrak{s}(A(T)) := \uparrow^{n-1} A(T) \otimes \mathrm{sgn}_n,$$

there is a Σ_n-equivariant isomorphism

$$\mathfrak{s}(A(T)) \cong (\mathfrak{s}A)(T)$$

which commutes with morphisms in dg-$\mathtt{Col}_K$.

Proof. Denote by $\widehat{\mathrm{sgn}}_p$ a basis element for the one-dimensional signum representation of Σ_p :

$$\sigma \cdot \widehat{\mathrm{sgn}}_p := \mathrm{sgn}_p(\sigma)\widehat{\mathrm{sgn}}_p. \tag{3.24}$$

Define a dg-$\mathtt{Mod}_{\mathbf{k}}$ morphism $\Psi_{A,T} : \mathfrak{s}(A\langle T\rangle) \longrightarrow (\mathfrak{s}A)\langle T\rangle$, by

$$\begin{aligned}\Psi_{A,T}(\uparrow^{n-1} (\alpha_0 \otimes \cdots \otimes \alpha_k) \otimes \widehat{\mathrm{sgn}}_n) &:= \\ (-1)^{\sum_{j=1}^k (m_0+\ldots+m_j-j)\alpha_j}(\uparrow^{m_0-1} \alpha_0 \otimes \widehat{\mathrm{sgn}}_{m_0})\otimes &\quad \cdots \quad \otimes(\uparrow^{m_k-1} \alpha_k \otimes \widehat{\mathrm{sgn}}_{m_k}),\end{aligned}$$

where $k = |T|$ and $m_j = \mathfrak{n}(v)$. We will prove that $\Psi_{A,T}$ is an isomorphism equivariant with respect to the right action of $\Sigma(T)$. From the definition it is clear that the following diagram commutes for any morphism $\xi : A \to B$ of collections:

$$\begin{array}{ccc} \mathfrak{s}(A\langle T\rangle) & \xrightarrow{\Psi_{A,T}} & (\mathfrak{s}A)\langle T\rangle \\ {\scriptstyle \mathfrak{s}(\xi\langle T\rangle)}\big\downarrow & & \big\downarrow{\scriptstyle (\mathfrak{s}(\xi))\langle T\rangle} \\ \mathfrak{s}(B\langle T\rangle) & \xrightarrow{\Psi_{B,T}} & (\mathfrak{s}B)\langle T\rangle \end{array} .$$

The proof is by induction on the maximal distance of a leaf from the root in the tree T. The root vertex is always labeled v_0. If the maximal distance is 1, then all the vertices $v_1, \ldots, v_k$ are connected to the root by an edge. By adding factors of $A(1)$ if necessary we can always assume that $m_0 = k = |T|$. Then it is easy to see that the subgroup $\Sigma(T)$ is the normalizer of the subgroup $\Sigma_{m_1} \times \cdots \times \Sigma_{m_k} \subset \Sigma_n$ and is generated by two types of elements. The first type of element is in the image of of the imbeddings $\Sigma_{m_i} \hookrightarrow \Sigma_n$,

$$\sigma \mapsto \hat\sigma, \quad \hat\sigma(m_1 + \cdots + m_{i-1} + j) := m_1 + \cdots + m_{i-1} + \sigma(j).$$

The second type of element transposes two adjacent intervals of integers of the same length m_i:

$$\begin{aligned} \varphi(m_0 + \cdots + m_{i-1} + j) &:= m_0 + \cdots + m_i + j, \\ \varphi(m_0 + \cdots + m_i + j) &:= m_0 + \cdots + m_{i-1} + j, \text{for } 1 \le j \le m_i, \text{ and} \\ \varphi(j) &:= j \text{ for all other values of } j. \end{aligned}$$

The elements $\hat\sigma$ of the first type act on the tensor product $A(k) \otimes A(m_1) \otimes \cdots \otimes A(m_k)$ according to the action of Σ_{m_i} in the i-th factor (counting $A(k)$ as the 0-th factor). The elements φ of the second type act by transposing adjacent factors $A(m_i) = A(m_{i+1})$ while at the same time acting on $A(k)$ as the transposition of i and $i+1$. Clearly $\Psi_{A,T}$ is equivariant with respect to the action of the first type of elements, since

$$\mathrm{sgn}_n|_{\Sigma_{m_i}} = \mathrm{sgn}_{m_i}.$$

We need only verify that it is equivariant with respect to the action of the second type. Therefore, assuming that $m_i = m_{i+1}$, let φ be the transposition of intervals i and $i+1$ and $\tau = (i, i+1) \in \Sigma_k$ be the transposition of $i, i+1$, then

$$\begin{aligned}
&\Psi_{A,T}\left(\uparrow^{n-1}(\alpha_0 \otimes \cdots \otimes \alpha_i \otimes \alpha_{i+1} \otimes \cdots \otimes \alpha_k \otimes \widehat{\mathrm{sgn}}_n)\varphi\right) \\
&\quad = \Psi_{A,T}\left(\mathrm{sgn}_n(\varphi)(-1)^{\alpha_i\alpha_{i+1}} \uparrow^{n-1}(\alpha_0\tau \otimes \cdots \otimes \alpha_{i+1} \otimes \alpha_i \otimes \cdots \otimes \alpha_k \otimes \widehat{\mathrm{sgn}}_n)\right) \\
&\quad = (-1)^{\sum_{j=1}^{k}(k+m_1+\ldots+m_j-j)\alpha_j + (m_i-1)(\alpha_i+\alpha_{i+1}) + \alpha_i\alpha_{i+1} + m_i^2}(\uparrow^{k-1}\alpha_0\tau \otimes \widehat{\mathrm{sgn}}_k) \otimes \cdots \\
&\qquad \otimes(\uparrow^{m_i-1}\alpha_{i+1} \otimes \widehat{\mathrm{sgn}}_{m_i}) \otimes (\uparrow^{m_i-1}\alpha_i \otimes \widehat{\mathrm{sgn}}_{m_i}) \otimes \cdots \otimes (\uparrow^{m_k-1}\alpha_k \otimes \widehat{\mathrm{sgn}}_{m_k}).
\end{aligned}$$

The new terms in the exponent of the coefficient come from the substitution $\mathrm{sgn}_n(\varphi) = (-1)^{m_i^2}$ and the change in order of α_i and α_{i+1}. On the other hand

$$\begin{aligned}
&\left(\Psi_{A,T}\left(\uparrow^{n-1}(\alpha_0 \otimes \cdots \otimes \alpha_i \otimes \alpha_{i+1} \otimes \cdots \otimes \alpha_k \otimes \widehat{\mathrm{sgn}}_n)\right)\right)\varphi \\
&\quad = \Big((-1)^{\sum_{j=1}^{k}(k+m_1+\ldots+m_j-j)\alpha_j}(\uparrow^{k-1}\alpha_0 \otimes \widehat{\mathrm{sgn}}_k) \otimes \cdots \\
&\qquad \otimes(\uparrow^{m_i-1}\alpha_{i+1} \otimes \widehat{\mathrm{sgn}}_{m_i}) \otimes (\uparrow^{m_i-1}\alpha_i \otimes \widehat{\mathrm{sgn}}_{m_i}) \otimes \cdots \otimes (\uparrow^{m_k-1}\alpha_k \otimes \widehat{\mathrm{sgn}}_{m_k})\Big)\varphi \\
&\quad = (-1)^{\sum_{j=1}^{k}(k+m_1+\ldots+m_j-j)\alpha_j + (\alpha_i+m_i-1)(\alpha_{i+1}+m_i-1)}(\uparrow^{k-1}\alpha_0 \otimes \widehat{\mathrm{sgn}}_k)\tau \otimes \cdots \\
&\qquad \otimes(\uparrow^{m_i-1}\alpha_{i+1} \otimes \widehat{\mathrm{sgn}}_{m_i}) \otimes (\uparrow^{m_i-1}\alpha_i \otimes \widehat{\mathrm{sgn}}_{m_i}) \otimes \cdots \otimes (\uparrow^{m_k-1}\alpha_k \otimes \widehat{\mathrm{sgn}}_{m_k}) \\
&\quad = (-1)^{\sum_{j=1}^{k}(k+m_1+\ldots+m_j-j)\alpha_j + (\alpha_i+m_i-1)(\alpha_{i+1}+m_i-1)+1}(\uparrow^{k-1}\alpha_0\tau \otimes \widehat{\mathrm{sgn}}_k) \otimes \cdots \\
&\qquad \otimes(\uparrow^{m_i-1}\alpha_{i+1} \otimes \widehat{\mathrm{sgn}}_{m_i}) \otimes (\uparrow^{m_i-1}\alpha_i \otimes \widehat{\mathrm{sgn}}_{m_i}) \otimes \cdots \otimes (\uparrow^{m_k-1}\alpha_k \otimes \widehat{\mathrm{sgn}}_{m_k}),
\end{aligned}$$

so $\Psi_{A,T}$ and φ commute.

Suppose that we have proved the first part of the Lemma for trees with a maximal distance from leaf to root equal to m and we want to prove it for trees with maximal distance $m+1$. Any such tree T is the result of grafting trees T_1 to T_k to a corolla T_0 with k leaves where the maximal distance for from leaf to root in all the T_i for $i \geq 1$ is less than or equal to m and is equal to m for at least one such T_i. There is a dg-$\mathtt{Mod}_{\mathbf{k}}$ isomorphism

$$A\langle T\rangle \cong A(k) \otimes A\langle T_1\rangle \otimes \cdots \otimes A\langle T_k\rangle.$$

The group $\Sigma(T)$ can be identified with the normalizer of $\Sigma(T_1) \times \cdots \times \Sigma(T_k)$ in Σ_n. Then $\Psi_{A,T}$ can be factored into the composition of two morphisms

$$\Psi'_{A,T} : \mathfrak{s}(A\langle T\rangle) \longrightarrow (\mathfrak{s}A)(k) \otimes \mathfrak{s}(A\langle T_1\rangle) \otimes \cdots \otimes \mathfrak{s}(A\langle T_k\rangle),$$

followed by $1\!\!1 \otimes \Psi_{A,T_1} \otimes \cdots \otimes \Psi_{A,T_k}$. The equivariance of $\Psi'_{A,T}$ is proven by an argument analogous to the first step of the induction. The equivariance of the Ψ_{A,T_i} follows from the induction hypothesis. Therefore the composition $\Psi_{A,T}$ is $\Sigma(T)$ equivariant.

The isomorphism in the second half of the Lemma depends on the choice of a planar tree T^{pl} representing the abstract tree T and the isomorphism

$$\theta_{A,T^{\mathrm{pl}}} : A(T) \xrightarrow{\cong} A\langle T^{\mathrm{pl}}\rangle \otimes_{\mathbf{k}[\Sigma(T^{\mathrm{pl}})]} \mathbf{k}[\Sigma_n].$$

Define a dg-$\mathtt{Mod}_{\mathbf{k}}$ morphism

$$\Lambda_{T^{\mathrm{pl}}} : \left(\uparrow^{n-1} A\langle T^{\mathrm{pl}}\rangle \otimes_{\mathbf{k}[\Sigma(T^{\mathrm{pl}})]} \mathbf{k}[\Sigma_n]\right) \otimes \mathrm{sgn}_n \longrightarrow \left(\uparrow^{n-1} A\langle T^{\mathrm{pl}}\rangle \otimes \mathrm{sgn}_n\right) \otimes_{\mathbf{k}[\Sigma(T^{\mathrm{pl}})]} \mathbf{k}[\Sigma_n]$$

by

$$\Lambda_{T^{\mathrm{pl}}} : (u \otimes g) \otimes \widehat{\mathrm{sgn}}_n \longmapsto \mathrm{sgn}_n(g)(u \otimes \widehat{\mathrm{sgn}}_n) \otimes g, \text{ for } u \in \uparrow^{n-1} A\langle T^{\mathrm{pl}}\rangle.$$

One checks immediately that $\Lambda_{T^{\mathrm{pl}}}$ is well-defined with respect to the relations for the tensor product over $k[\Sigma(T^{\mathrm{pl}})]$ and is Σ_n equivariant.

Since $\Psi_{A,T^{\mathrm{pl}}}$ is $\Sigma(T^{\mathrm{pl}})$-equivariant the tensor product $\Psi_{A,T^{\mathrm{pl}}} \otimes id$ is well-defined on $\mathfrak{s}(A\langle T^{\mathrm{pl}}\rangle) \otimes_{k[\Sigma(T^{\mathrm{pl}})]} \mathbf{k}[\Sigma_n]$ and clearly right Σ_n-equivariant. Then define

$$\Phi_{A,T} := \theta^{-1}_{\mathfrak{s}A,T^{\mathrm{pl}}} \circ (\Psi_{A,T^{\mathrm{pl}}} \otimes id) \circ \Lambda_{T^{\mathrm{pl}}} \circ \theta_{A,T^{\mathrm{pl}}}.$$

Clearly, the following diagram commutes

$$\begin{tikzcd}
\mathfrak{s}(A(T)) \arrow[r, "\Phi_{A,T}"] \arrow[d, "\mathfrak{s}(\xi(T\rangle)"'] & (\mathfrak{s}A)(T\rangle \arrow[d, "(\mathfrak{s}(\xi))(T)"] \\
\mathfrak{s}(B(T)) \arrow[r, "\Phi_{B,T}"] & (\mathfrak{s}B)(T)
\end{tikzcd} .$$

and therefore $\Phi_{-,T} : \mathfrak{s}(-(T)) \longrightarrow (\mathfrak{s}-)(T)$ defines a natural transformation of functors on the category dg-$\mathtt{Col}_K$. ■

Proof of the Proposition. Define

$$\Phi_A := \bigoplus_{T \in \mathtt{Tree}^{\mathrm{pl}}_n} \Phi_{A,T} : \mathfrak{s}(\Gamma(A))(n) \longrightarrow \Gamma(\mathfrak{s}A)(n).$$

The explicit form of Φ_A depends on the description of $\Gamma(A)(n)$ in (3.23) as a sum indexed by the set of planar trees $\mathtt{Tree}^{\mathrm{pl}}_n$, but for any fixed choice of indexing set $\mathtt{Tree}^{\mathrm{pl}}_n$, one planar tree representing each abstract tree, Φ defines a natural transformation since for each T $\Phi_{-,T}$ is a natural transfomation. The verification that Φ_A is a morphism of operads (i.e., is compatible with operad composition) and hence that Φ is a natural transformation of functors from dg-$\mathtt{Col}_K$ to dg-$\mathtt{Op}_K$ will be left to the industrious reader. ■

3.2. Quadratic operads and Koszul operads.

The definition of a quadratic operad is based on a fairly straightforward analogy with the definition of a quadratic algebra as the quotient of a free associative algebra by an ideal of quadratic relations. We have already defined free operads so it remains only to define ideals in an operad. First we introduce the concept of a module over an operad, see [16]. Henceforth, unless otherwise indicated, operad means K-operad for some fixed $\mathbf{k}$-algebra K.

Definition 3.2.1. *A* left-module over an operad *A is a collection $M = \{M(n);\ n \geq 1\}$ together with maps*

$$\lambda : \bigoplus_{1 \leq i \leq l} \Big\{ A(l) \otimes A(m_1) \otimes \cdots \otimes M(m_i) \otimes \cdots \otimes A(m_l) \Big\} \longrightarrow M(m_1 + \cdots + m_l)$$

for any $m_1, \ldots, m_l \geq 1$ satisfying the operadic form of the standard axioms for a left module over an algebra.

A right-module over an operad *A is a collection $M = \{M(n);\ n \geq 1\}$ together with maps*

$$\rho : M(l) \otimes A(m_1) \otimes \cdots \otimes A(m_l) \longrightarrow M(m_1 + \cdots + m_l)$$

for any $m_1, \ldots, m_l \geq 1$ satisfying the corresponding operadic form of the identities for a right module.

We can also combine the maps appearing in these definitions using the $\square$-product introduced in Section 1.1. For example, a right-A-module structure is given by a morphism

$$\rho : M \square A \longrightarrow M$$

satisfying the standard identity between the composite maps

$$\rho \circ (\rho \otimes \mathbb{1}) = \rho \circ (\mathbb{1} \otimes \mu_A) \circ a_{M,A,A} : (M \square A) \square A \longrightarrow M, \tag{3.25}$$

where $a_{M,A,A}$ is the associativity constraint for the $\square$-product of collections. The analogous description of a left-A-module uses the following definition.

Definition 3.2.2. *Given three collections A, B, C, define the* relative $\square$-product $A \square'(B, C)$ *to be the subcollection of the $\square$-product of A with the collection $B \oplus C$ consisting of terms with just one $\otimes$ factor from C. The component of arity m is*

$$A \square'(B, C)(m) := \bigoplus_{\substack{1 \leq i \leq q \leq m, \\ m_1 + \cdots + m_q = m}} A(q) \otimes \big(B(m_1) \otimes \cdots \otimes B(m_{i-1}) \otimes C(m_i) \otimes B(m_{i+1}) \cdots \otimes B(m_q)\big).$$

The associativity constraint for the $\square$-product of collections defines an associativity constraint for the relative $\square$-product,

$$a'_{A,B,(C,D)} : A\square'(B\square C, B\square'(C,D)) \xrightarrow{\cong} (A\square B)\square'(C,D). \tag{3.26}$$

Using the relative $\square$-product, we can define a left-A-module structure on a collection M as a morphism

$$\lambda : A\square'(A,M) \longrightarrow M,$$

satisfying an identity analogous to equation (3.25).

Definition 3.2.3. *An A-module M is a collection with right and left A-module structures, which are compatible in the obvious sense.*

Equivalently, left and right A modules can be described in terms of the left and right $\circ_i$-operations:

$$\circ_i^\lambda = \circ_i : A(n) \otimes M(m) \to M(m+n-1), \quad \circ_i^\rho = \circ_i : M(n) \otimes A(m) \to M(n+m-1).$$

The operad A itself is an A-module. If $\gamma : A \to B$ is an operad map, then γ induces a A-module structure on B. The definition of an ideal in an operad is derived from the definition of an A module in the standard way.

Definition 3.2.4. *An* ideal in an operad *A is a subcollection which also an A-submodule.*

The forgetful functor $U : \mathtt{Mod}_A \to \mathtt{Col}_K$ from the category of A-modules to the category of collections has a left adjoint $A\langle - \rangle : \mathtt{Col} \to \mathtt{Mod}_A$, and $A\langle B\rangle$ is called the *free A-module* on the collection B. It is defined by

$$A\langle B\rangle := A\square'(\mathbb{1}, B\square A),$$

where $\mathbb{1}$ is the unit collection defined in (1.43) and the structure maps are defined by applying the associativity constraint to regroup terms followed by operad composition in A. For example the unit morphism together with the relative associativity constraint (3.26) defines an isomorphism

$$A\square'(A, A\square'(\mathbb{1}, B\square A)) \cong A\square'(A\square\mathbb{1}, A\square'(\mathbb{1}, B\square A)) \xrightarrow{a_{A,A,(I,B\square A)}} (A\square A)\square'(\mathbb{1}, B\square A),$$

which composed with the operad composition $A\square A \longrightarrow A$, defines the left A-module morphism:

$$A\square'(A, A\langle B\rangle) \longrightarrow A\langle B\rangle.$$

Definition 3.2.5. *Let A be an operad and $R \subset A$ a subcollection of A. The* operadic ideal generated by *R* in *A is the image in A of the free A-module $A\langle R\rangle$ under the morphism $A\langle R\rangle \longrightarrow A$ induced by the inclusion $R \hookrightarrow A$.*

Each K-operad A can be represented as a quotient $\Gamma(E)/(R)$, where E and R are K-collections and (R) is the operadic ideal generated by R in $\Gamma(E)$; we write $A = \langle E; R \rangle$.

Definition 3.2.6. *A K-collection E such that $E(n) = 0$ if $n \neq 2$ is called a quadratic K-collection. An operad is called a* quadratic K-operad *if it is a quotient of the free operad on a quadratic K-collection modulo an ideal generated by a subspace $R \subset \Gamma(E)(3)$ of* defining relations.

Remark 3.2.7. Identities such as $\alpha = \alpha \cdot \sigma$ or $\alpha = -\alpha \cdot \sigma$ for a basis element $\alpha \in E(2)$ are not considered as defining relations, but rather as a description of the decomposition of $E(2)$ into a sum of copies of the trivial representation of Σ_2 and the sign representation. These identities in turn determine the decomposition of the space R as a representation of Σ_3, as we explain below. We want to point out that the free operad $\Gamma(E)$ generated by a quadratic collection E has the special feature that in (1.4.8) we need only consider binary trees.

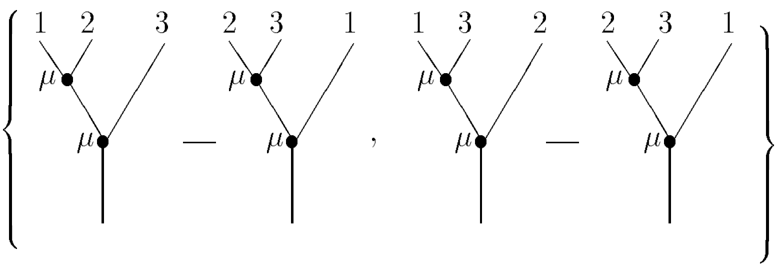
In a similar way we could define operads of type $(k, 2k-1)$ which are generated by a collection E satisfying $E(n) = 0$ for $n \neq k$ and have defining relations $R \subset \Gamma(E)(2k-1)$. In this case the free operad $\Gamma(E)$ would be described by trees having all vertices with arity k.

Example 3.2.8. The commutative associative operad *Com* is generated by the $\mathbf{k}$-collection

$$E_{Com}(n) = \begin{cases} \mathbf{k} \cdot \mu & \text{if } n = 2, \\ 0 & \text{if } n \neq 2, \end{cases}$$

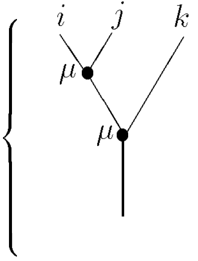
where $\mathbf{k} \cdot \mu$ is the trivial representation of Σ_2. The ideal of relations is generated by:

$R_{Com} :=$ Span



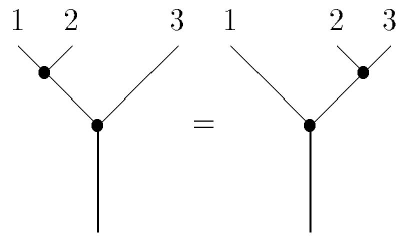
which is a Σ_3-invariant subspace of

$\Gamma(E_{Com})(3) =$ Span



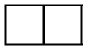
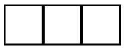
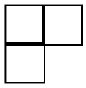
$|(i,j,k) \in \{(1,2,3),(1,3,2),(2,3,1)\}$

We could also say that R_{Com} is the subspace generated over Σ_3 by the 'associativity'



which is perhaps the most natural way to introduce this operad. We could also give a more formal definition as $\Gamma(\mu)/(\mu \circ_1 \mu - \mu \circ_2 \mu)$.

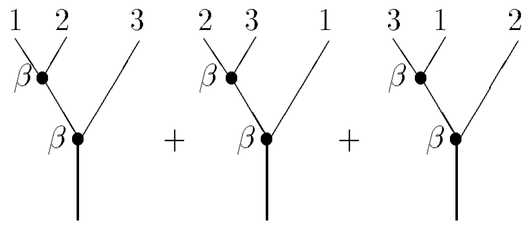
In terms of the representation theory of the symmetric groups E_{Com} has Young diagram  and $\Gamma(E_{Com})(3)$ is the corresponding induced representation of Σ_3 which is 3-dimensional and decomposes as the sum of the 1-dimensional trivial representation isomorphic to $Com(3)$ and the 2-dimensional irreducible representation given by the relations R_{Com}. Expressed in Young diagrams:

$\Gamma(E_{Com})(3) \cong \Gamma($  $)(3) \cong$  $\bigoplus$  $\cong Com(3) \bigoplus R_{Com}$

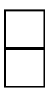
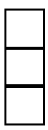
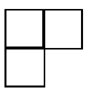
Example 3.2.9. The Lie operad *Lie* is generated by the **k**-collection

$$E_{Lie}(n) = \begin{cases} \mathbf{k} \cdot \beta & \text{if } n = 2 \\ 0 & \text{if } n \neq 2, \end{cases}$$

where $\mathbf{k} \cdot \beta$ is the signum representation of Σ_2. The ideal of relations is generated by the Jacobi identity:

$R_{Lie} :=$ Span 

In this case the generator E_{Lie} is the signum representation of Σ_2 with Young diagram Young diagram  and the corresponding induced represetation of Σ_3 is again 3-dimensional, but this time it decomposes into the sum of the 1-dimensional signum representation given by the subspace of relations and the complementary 2-dimensional irreducible representation isomorphic to $Lie(3)$:

$\Gamma(E_{Lie})(3) \cong \Gamma($  $)(3) \cong$  $\bigoplus$  $\cong R_{Lie} \bigoplus Lie(3)$

Example 3.2.10. For the associative operad $E(2)$ is the regular representation of Σ_2 with basis $\{\alpha, \alpha\cdot\tau\}$. Then $\Gamma(E_{Ass})(3)$ is two copies of the regular representation

$$\Gamma(E_{Ass})(3) = \mathrm{Span}\{(\alpha\otimes_1\alpha)\otimes(i,j,k),\ (\alpha\otimes_2\alpha)\otimes(i,j,k)\}.$$

The defining relations are

$$R_{Ass} := \mathrm{Span}\{(\alpha\otimes_1\alpha - \alpha\otimes_2\alpha)\otimes(i,j,k)\}.$$

In the next definition we introduce a duality which is central to the concept of Koszul operads.

Definition 3.2.11. *Given a K-collection E, define the dual as the K^{op}-collection $E^{\#} = \{E^{\#}(n)\}_{n\geq 1}$ with $E^{\#}(n) := Hom_{\mathbf{k}}(E(n),\mathbf{k})$. The Σ_n representation on $E(n)$ determines a dual representation on $E^{\#}(n)$ by*

$$\langle\lambda\cdot\sigma,\alpha\rangle := \langle\lambda,\alpha\cdot\sigma^{-1}\rangle,\ \textit{for } \lambda\in E^{\#}(n),\ \sigma\in\Sigma_n \textit{ and } \alpha\in E(n).$$

The Czech dual *is the K^{op}-collection $E^{\vee} = \{E^{\vee}(n)\}_{n\geq 1}$ with $E^{\vee}(n) := E^{\#}(n)\otimes\mathrm{sgn}_n$. This is equivalent to defining $E^{\vee}(n)$ as the dual space with Σ_n representation*

$$\langle\lambda\cdot\sigma,\alpha\rangle := \mathrm{sgn}_n(\sigma)\langle\lambda,\alpha\cdot\sigma^{-1}\rangle. \tag{3.27}$$

There is a very important construction in the theory of operads called the *quadratic dual operad*, defined as a quotient of $\Gamma(E^{\vee})$ by certain relations and related to $\mathbf{D}(A)$ by a dg-$\mathtt{Mod}_{\mathbf{k}}$ morphism. The full definition is given below. In order to describe these relations we need to extend the pairing between $E(n)$ and $E^{\vee}(n)$ to a pairing between $\Gamma(E)(n)$ and $\Gamma(E^{\vee})(n)$. The first condition defining the extension is that the components $E(T)$ and $E^{\vee}(T')$ are orthogonal if $T\neq T'$ (as an abstract tree). For each isomorphism class of trees we choose a planar tree with vertices in planar order $(v_1,\cdots,v_p)$ and represent elements of the unordered tensor product $E(T)$ and $E^{\vee}(T)$ as products $\alpha_1\otimes\cdots\alpha_p$ and $\lambda_1\otimes\cdots\otimes\lambda_p$ respectively. Then we define

$$\begin{aligned}\langle(\lambda_1\otimes\cdots\otimes\lambda_p)\otimes(i_1,\cdots,i_n),(\alpha_1\otimes\cdots\otimes\alpha_p)\otimes(i_1,\cdots,i_n)\rangle &=\\ = (-1)^{\alpha_1(\lambda_2+\cdots+\lambda_p)+\alpha_2(\lambda_3+\cdots+\lambda_p)+\cdots+\alpha_{p-1}\lambda_p}\mathrm{sgn}_n(i_1,\cdots,i_n)\langle\lambda_1,\alpha_1\rangle\cdots\langle\lambda_p,\alpha_p\rangle\end{aligned} \tag{3.28}$$

For example, the pairing between $\Gamma(E^{\vee})(3)$ and $\Gamma(E)(3)$ is defined by

$$\begin{aligned}&\langle(\lambda\otimes_1\lambda')\otimes(i,j,k),(\alpha\otimes_1\alpha')\otimes(i',j',k')\rangle :=\\ &:= \begin{cases}(-1)^{\lambda'\alpha}\mathrm{sgn}_3(i,j,k)\langle\lambda,\alpha\rangle\langle\lambda',\alpha'\rangle & \text{if } (i,j,k)=(i',j',k'),\\ (-1)^{\lambda'\alpha}\mathrm{sgn}_3(i,j,k)\langle\lambda,\alpha\rangle\langle\lambda',\alpha'\cdot\tau\rangle & \text{if } (j,i,k)=(i',j',k'),\\ 0 & \text{otherwise.}\end{cases}\end{aligned} \tag{3.29}$$

In this example we use the product $\otimes_1$ defined in (3.15).

Definition 3.2.12. *The* quadratic dual *of a quadratic operad* $A = \langle E; R\rangle$ *is* $A^! := \langle E^\vee; R^\perp\rangle$, *where* $R^\perp \subseteq \Gamma(E^\vee)(3)$ *is the orthogonal complement to (i.e., the annihilator of) the relations* $R \subseteq \Gamma(E)(3)$ *defining* A.

It is an instructive exercise to show that $(-)^!$ is an involution, $(A^!)^! \cong A$.

Example 3.2.13. The quadratic dual of the commutative operad is generated by

$$E^\vee_{Com} = \yng(2)^{\vee} = \yng(1,1) = E_{Lie}.$$

The space of defining relations for $Com^!$ is the annihilator in $\Gamma(E_{Lie})(3)$ (relative to the pairing (3.29)) of the defining relations for the commutative operad. Since the annihilator is a subrepresentation of complementary dimension, it is the one dimensional subrepresentation given by R_{Lie}:

$$R_{Com^!} = R^\perp_{Com} = \yng(2,1)^{\perp} = \yng(1,1,1) = R_{Lie}.$$

Therefore Lie is the quadratic dual of Com and conversely:

$$Com^! = Lie, \quad Lie^! = Com.$$

In describing the associative operad we use a basis for $\Gamma(E)(3)$ involving both terms with $\otimes_1$ and $\otimes_2$. In this case we need to use the identities (3.15) to reduce to terms of type $\otimes_1$ and apply equations (3.29). For example, the pairing between two expressions involving $\otimes_2$ is given by:

$$\begin{aligned}\langle(\lambda \otimes_2 \lambda')\otimes(i,j,k) \quad &, \quad (\alpha\otimes_2\alpha')\otimes(i',j',k')\rangle \\ &= \langle(\lambda\cdot\tau\otimes_1\lambda')\otimes(j,k,i),(\alpha\cdot\tau\otimes_1\alpha')\otimes(j',k',i')\rangle \\ &= -\langle(\lambda\otimes_1\lambda')\otimes(j,k,i),(\alpha\otimes_1\alpha')\otimes(j',k',i')\rangle\end{aligned} \tag{3.30}$$

The minus sign appears because of the identity

$$\langle\lambda\cdot\tau,\alpha\cdot\tau\rangle = \mathrm{sgn}(\tau)\langle\lambda,\alpha\rangle = -\langle\lambda,\alpha\rangle$$

in the definition of the Czech dual.

In the associative operad, $E_{Ass}(2)$ is the regular representation of Σ_2 and so is $E^\vee_{Ass}(2)$. We denote the basis of $E^\vee_{Ass}(2)$ by $\{\lambda, \lambda\cdot\tau\}$. From the definition of the pairing in equation (3.30), it is clear that

$$R^\perp_{Ass} = \mathrm{Span}\{(\lambda\otimes_1\lambda - \lambda\otimes_2\lambda)\otimes(i,j,k)\}$$

and thus

$$Ass^! = Ass.$$

Theorem 3.2.14. *For a quadratic K-operad A there is a natural transformation of functors*

$$\Theta_A : \mathbf{D}(A) \to A^!,$$

inducing an isomorphism

$$H^0(\mathbf{D}(A)(n), \delta) \cong A^!(n), \text{ for each } n \geq 2.$$

Remark 3.2.15. Note that the '0' in H^0 refers to the "tree-degree" not the total degree, and that $\mathbf{D}(A)$ is negatively graded.

Proof. By definition

$$\begin{aligned}
\mathbf{D}(A)(n)^0 &= \bigoplus_{\text{binary } T \in \mathtt{Tree}_n} \mathfrak{s}^{-1}(A^{\#}(T) \otimes \det(T)) \\
&\cong \bigoplus_{\text{binary } T \in \mathtt{Tree}_n} \mathfrak{s}^{-1}(\uparrow A^{\#}(T)) \text{ by Lemma 3.1.9} \\
&\cong \bigoplus_{\text{binary } T \in \mathtt{Tree}_n} (\mathfrak{s}^{-1}(\uparrow A^{\#}))(T)) \text{ by Lemma 3.1.20} \\
&\cong \bigoplus_{\text{binary } T \in \mathtt{Tree}_n} (A^{\vee})(T).
\end{aligned}$$

The last isomorphism comes from the fact that the trees are binary and

$$(\mathfrak{s}^{-1}(\uparrow A^{\#}))(2) \cong \downarrow (\uparrow A^{\#}))(2) \otimes \mathrm{sgn}_2 \cong A^{\#}(2) \otimes \mathrm{sgn}_2 = A^{\vee}(2).$$

Thus $\mathbf{D}(A)^0(n) \cong \Gamma(A^{\vee}(2))(n)$.

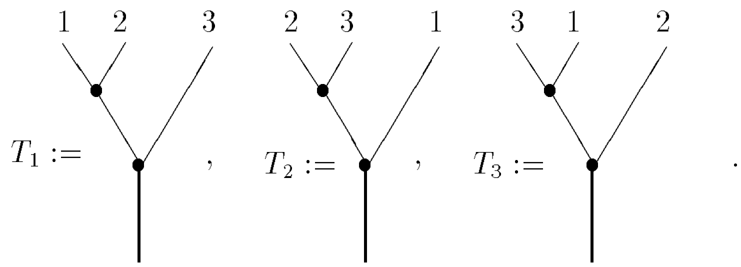
The morphism Θ_A is defined by composing with the projection π onto $A^!(n)$:

$$\Theta_A : \mathbf{D}(A)^0(n) \xrightarrow{\varphi} \Gamma(A^{\vee}(2))(n) \xrightarrow{\pi} \big(\Gamma(A^{\vee}(2))(n)/(R_A^{\perp})(n)\big) = A^!(n).$$

Finally we show that $\delta(\mathbf{D}(A)(n)^{-1}) = Ker(\Theta_A)$, proving that there is an isomorphism in top dimensional cohomology:

$$\overline{\Theta}_A : H^0(\mathbf{D}(A)(n), \delta) = \mathbf{D}(A)(n)^0/\delta(\mathbf{D}(A)(n)^{-1}) \xrightarrow{\cong} A^!(n).$$

The trees indexing $\mathbf{D}(A)(n)^{-1}$ are almost binary with one distinguished vertex having 3 incoming edges and all other vertices binary. For any such tree denote the distinguished vertex by v_T. The differential δ acts nontrivially only on the coefficient from $A^{\#}(3)$ at v_T. The image $\delta(A^{\#}(T) \otimes \mathrm{Det}(T))$ is contained in the direct sum of three terms labeled by the new binary trees created by grafting at the vertex v_T one of the following subtrees:



The space of defining relations R_A for a quadratic operad A appear as the left-hand term in an exact sequence:

$$0 \longrightarrow R_A \xrightarrow{i} \bigoplus_{i=1,2,3} A(T_i) \xrightarrow{\circ_x} A(3) \longrightarrow 0. \tag{3.31}$$

The middle term is $\Gamma(A(2))(3)$, i is the inclusion and $\circ_x$ contracts the unique internal edge of the trees T_i. Dualizing, we get the exact sequence

$$0 \longleftarrow R_A^{\#} \xleftarrow{i^{\#}} \bigoplus_{i=1,2,3} A^{\#}(T_i) \xleftarrow{\delta} A^{\#}(3) \longleftarrow 0. \tag{3.32}$$

Therefore $\delta(A^{\#}(3)) = Ker(i^{\#}) = R_A^{\perp}$. For each tree T in the indexing set of $\mathbf{D}(A)(n)^{-1}$ take the tensor product of the sequence (3.32) with the tensor product of $A^{\#}(2)$ over the binary vertices of T. We can take the sum of the δ maps of all these sequence inside the complex $\mathbf{D}(A)(n)$ giving

$$\delta(\mathbf{D}(A)(n)^{-1}) = \varphi^{-1}(R_A^{\perp}) = Ker(\Theta_A). \qquad \blacksquare$$

Definition 3.2.16. *A dg-operad A is called a* Koszul operad *if Θ_A of Theorem 3.2.14 is a quasi-isomorphism; that is, for each n the complexes $\mathbf{D}(A)(n)^*$ are exact everywhere except in degree* 0*, hence*

$$H^*(\mathbf{D}(A)(n), \delta) = H^0(\mathbf{D}(A)(n), \delta) \cong A^!(n).$$

There is another characterization of a Koszul operad based on the Koszul complexes, which are defined below by analogy to the theory of quadratic algebras. For the concept of a Koszul quadratic algebra see [99]. In that case, the module underlying the Koszul complex is the quadratic dual algebra tensored with the dual to the original algebra. For example, in the case of the symmetric algebra, where the quadratic dual algebra equals the exterior algebra, the Koszul complex as a module is the tensor product of the symmetric and exterior algebras, and as a complex is isomorphic to the deRham complex for polynomial differential forms.

Definition 3.2.17. *The* Koszul complex *of a quadratic operad A has as component in degree p and arity n:*

$$\mathbf{K}(A^!)(n)^p := A^!(p) \otimes_{K^{\mathrm{op}}[p,p]} (\mathrm{sgn}_p \otimes A^\#[p,n]).$$

The differential $\delta^p_{\mathbf{K}}$ is the composition (deleting subscripts in the tensor products for ease in reading):

$$\begin{aligned} A^!(p) \otimes (\mathrm{sgn}_p \otimes A^\#[p,n]) \quad \longrightarrow \quad & A^!(p) \otimes (\mathrm{sgn}_p \otimes (A^\#[p,p+1] \otimes A^\#[p+1,n])) \\ \cong \; & A^!(p) \otimes (A^![p,p+1] \otimes \mathrm{sgn}_{p+1}) \otimes (A^\#[p+1,n]) \\ \cong \; & \big(A^!(p) \otimes A^![p,p+1]\big) \otimes (\mathrm{sgn}_{p+1} \otimes A^\#[p+1,n]) \\ \longrightarrow \; & A^!(p+1) \otimes (\mathrm{sgn}_{p+1} \otimes A^\#[p+1,n]). \end{aligned}$$

The first arrow is $1 \otimes 1 \otimes \tilde{\Delta} \otimes 1$, where $\tilde{\Delta} : A^\#[p,n] \to A^\#[p,p+1] \otimes A^\#[p+1,n]$ is the projection of the dual of the operadic multiplication $\Delta : A^\#[p,n] \to \bigoplus_{p<r<n} A^\#[p,r] \otimes A^\#[r,n]$ on the component with $r = p+1$. The isomorphism between the second and the third terms is explained in Lemma 3.2.19 below and the last arrow is composition in $A^!$.

There is an another characterization of the Koszul property based on the Koszul complex which will be useful is proving that *Com*, *Lie* and *Ass* are Koszul.

Theorem 3.2.18. *A quadratic operad A is Koszul if and only if the Koszul complex $K(A^!)(n)$ is exact for all $n \geq 2$.*

The proof of this theorem will be given in the next section. The rest of the section is devoted to the proof that the Koszul complex is well defined.

Lemma 3.2.19. *There is a natural isomorphism*

$$\mathrm{sgn}_p \otimes A^\#[p,p+1] \overset{\cong}{\rightarrow} A^![p,p+1] \otimes \mathrm{sgn}_{p+1}$$

of left $K^{\mathrm{op}}[p,p]$-right $K^{\mathrm{op}}[p+1,p+1]$-bimodules.

Proof. In the appendix to Section 1.1, we showed that $A^\#[p,p+1]$ has the structure of a representation, induced in two stages from $A^\#(1)^{\otimes p-1} \otimes A^\#(2)$ considered as a right Σ_2-module and a left Σ_{p-1}-module. More precisely, $A^\#(1)^{\otimes p-1} \otimes A^\#(2)$ has a right Σ_2-action on the last factor, and the induced representation

$$(A^\#(1)^{\otimes p-1} \otimes A^\#(2)) \otimes_{\mathbf{k}[\Sigma_2]} \mathbf{k}[\Sigma_{p+1}]$$

has a left Σ_{p-1}-action defined by

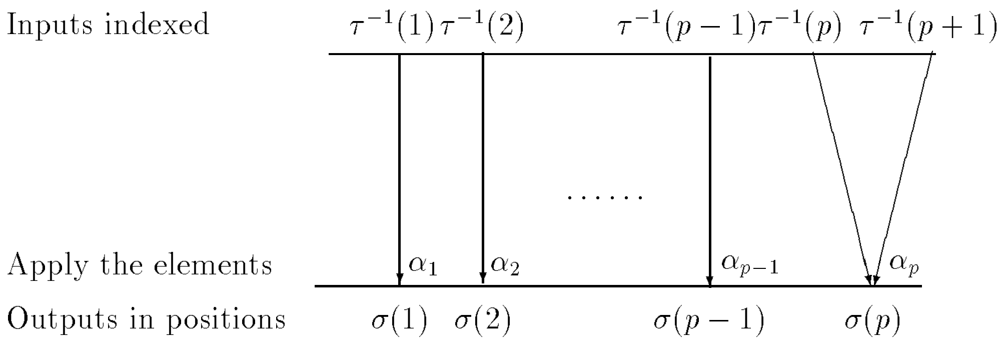
$$\rho \cdot (\alpha_1 \otimes \cdots \otimes \alpha_p \otimes \tau) = \alpha_{\rho^{-1}(1)} \otimes \cdots \otimes \alpha_{\rho^{-1}(p-1)} \otimes \alpha_p \otimes \tilde{\rho}\tau,$$

for $\rho \in \Sigma_{p-1}$, $\tau \in \Sigma_{p+1}$, $\alpha_1, \ldots, \alpha_{p-1} \in A^\#(1)$, $\alpha_p \in A^\#(2)$, and $\widetilde{\ } : \Sigma_{p-1} \to \Sigma_{p+1}$ is the natural inclusion.

Then

$$A^\#[p, p+1] \cong \mathbf{k}[\Sigma_p] \otimes_{\mathbf{k}[\Sigma_{p-1}]} \left((A^\#(1)^{\otimes p-1} \otimes A^\#(2)) \otimes_{\mathbf{k}[\Sigma_2]} \mathbf{k}[\Sigma_{p+1}] \right).$$

An element $\sigma \otimes (\alpha_1 \otimes \cdots \otimes \alpha_p) \otimes \tau \in A^\#[p, p+1]$, can be represented by the following picture with the top row indicating inputs and the bottom row outputs:



Let $V := A^\#(1)^{\otimes p-1} \otimes A^\#(2)$. Lemma 3.2.19 asserts the existence of an isomorphism

$$\begin{aligned} &\mathrm{sgn}_p \otimes \left(\mathbf{k}[\Sigma_p] \otimes_{\mathbf{k}[\Sigma_{p-1}]} \left[V \otimes_{\mathbf{k}[\Sigma_2]} \mathbf{k}[\Sigma_{p+1}] \right] \right) \longrightarrow \\ &\qquad\qquad \left(\mathbf{k}[\Sigma_p] \otimes_{\mathbf{k}[\Sigma_{p-1}]} \left[(V \otimes \mathrm{sgn}_2) \otimes_{\mathbf{k}[\Sigma_2]} \mathbf{k}[\Sigma_{p+1}] \right] \right) \otimes \mathrm{sgn}_{p+1}. \end{aligned}$$

In the second term $\mathbf{k}[\Sigma_2]$ acts diagonally on $V \otimes \mathrm{sgn}_2$. The isomorphism is defined by

$$\varphi : \widehat{\mathrm{sgn}}_p \otimes \sigma \otimes [v \otimes \tau] \longmapsto (\widehat{\mathrm{sgn}}_p(\sigma) \widehat{\mathrm{sgn}}_{p+1}(\tau)) \sigma \otimes [(v \otimes \widehat{\mathrm{sgn}}_2) \otimes \tau] \otimes \widehat{\mathrm{sgn}}_{p+1},$$

for $\sigma \in \Sigma_p$, $\tau \in \Sigma_{p+1}$, where $\widehat{\mathrm{sgn}}_p$ was defined in equation (3.24). Equivariance with respect to the left Σ_p-action follows from:

$$\begin{aligned} \varphi(\omega \cdot (\widehat{\mathrm{sgn}}_p \otimes \sigma \otimes [v \otimes \tau]) &= \varphi(\mathrm{sgn}_p(\omega)\ \widehat{\mathrm{sgn}}_p \otimes \omega\sigma \otimes [v \otimes \tau]) \\ &= (\mathrm{sgn}_p(\omega)\mathrm{sgn}_p(\omega\sigma)\mathrm{sgn}_{p+1}(\tau))\ \omega\sigma \otimes [(v \otimes \widehat{\mathrm{sgn}}_2) \otimes \tau] \otimes \widehat{\mathrm{sgn}}_{p+1} \\ &= (\mathrm{sgn}_p(\sigma)\mathrm{sgn}_{p+1}(\tau))\ \omega\sigma \otimes [(v \otimes \mathrm{sgn}_2) \otimes \tau] \otimes \widehat{\mathrm{sgn}}_{p+1} \\ &= \omega \left((\mathrm{sgn}_p(\sigma)\mathrm{sgn}_{p+1}(\tau))\ \sigma \otimes [(v \otimes \widehat{\mathrm{sgn}}_2) \otimes \tau] \otimes \widehat{\mathrm{sgn}}_{p+1} \right) \\ &= \omega \cdot \varphi((\widehat{\mathrm{sgn}}_p \otimes \sigma \otimes [v \otimes \tau]), \end{aligned}$$

for $\omega \in \Sigma_p$. Equivariance relative to the right Σ_{p+1}-action follows from a similar calculation.

The fact that this definition is compatible with the tensor product over $\mathbf{k}[\Sigma_{p-1}]$ between σ and $v \otimes \tau$ follows from the next equation (where $\widehat{\ } : \Sigma_{p-1} \to \Sigma_p$ and $\widetilde{\ } : \Sigma_{p-1} \to \Sigma_{p+1}$ are the natural inclusions):

$$\mathrm{sgn}_p(\sigma\hat{\rho})\mathrm{sgn}_{p+1}(\tau) = \mathrm{sgn}_{p-1}(\rho)\mathrm{sgn}_p(\sigma)\mathrm{sgn}_{p+1}(\tau) = \mathrm{sgn}_p(\sigma)\mathrm{sgn}_{p+1}(\tilde{\rho}\tau).$$

Compatibility with the tensor product over $\mathbf{k}[\Sigma_2]$ is shown as follows. Let ζ be the transposition in Σ_2 and $\bar{\zeta} \in \Sigma_{p+1}$ the transposition of the last 2 elements, then

$$\begin{aligned} \mathrm{sgn}_p \otimes \sigma \otimes [v \cdot \zeta \otimes \tau] &\mapsto \mathrm{sgn}_p(\sigma)\mathrm{sgn}_{p+1}(\tau)\,\sigma \otimes [(v \cdot \zeta \otimes \mathrm{sgn}_2) \otimes \tau] \otimes \mathrm{sgn}_{p+1} \\ \mathrm{sgn}_p \otimes \sigma \otimes [v \otimes \zeta\tau] &\mapsto \mathrm{sgn}_p(\sigma)\mathrm{sgn}_{p+1}(\zeta\tau)\,\sigma \otimes [(v \otimes \mathrm{sgn}_2) \otimes \bar{\zeta}\tau] \otimes \mathrm{sgn}_{p+1}, \end{aligned}$$

and both images are equal to $-\mathrm{sgn}_p(\sigma)\mathrm{sgn}_{p+1}(\tau)\sigma \otimes ([v \otimes \mathrm{sgn}_2) \cdot \zeta \otimes \tau] \otimes \mathrm{sgn}_{p+1}$. ∎

Lemma 3.2.20. *The Koszul operator $\delta_{\mathbf{K}}$ satisfies the condition for a differential $\delta_{\mathbf{K}}^2 = 0$.*

Proof. From Definition 3.2.17 to verify that $\delta_{\mathbf{K}}^2 = 0$ we look first at the map

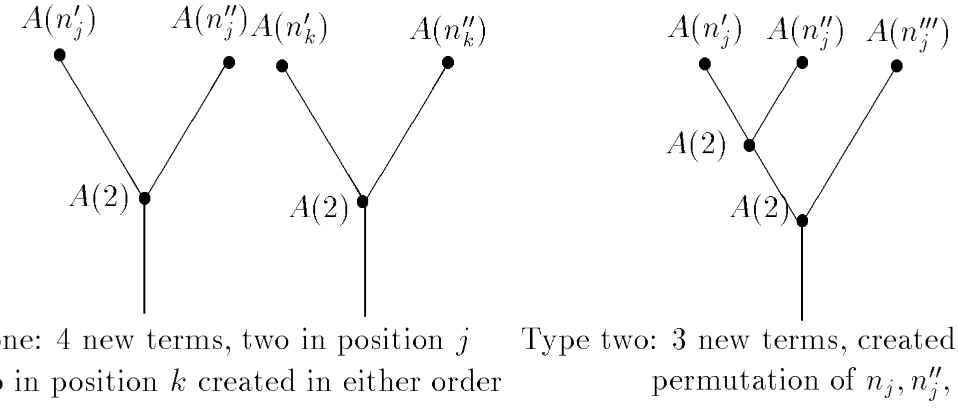
$$\tilde{\Delta}^2 := (id \otimes \tilde{\Delta}) \circ \tilde{\Delta} : A^{\#}[p,n] \longrightarrow A^{\#}[p,p+1] \otimes A^{\#}[p+1,p+2] \otimes A^{\#}[p+2,n].$$

The term $A^{\#}[p,p+1] \otimes A^{\#}[p+1,p+2]$ is a direct sum of components $A^{\#}[g] \otimes A^{\#}[f]$ indexed by pairs of surjections $[p+2] \xrightarrow{f} [p+1] \xrightarrow{g} [p]$. It is necessary to distinguish between two types of terms.

Terms of the first type arise from the dual map to the tensor product of a pair of disjoint binary compositions $A(2) \otimes A(n_j') \otimes A(n_j'') \to A(n_j' + n_j'')$ and $A(2) \otimes A(n_k') \otimes A(n_k'') \to A(n_k' + n_k'')$ and are indexed by pairs f, g such that $f^{-1}(g^{-1}(i))$ has cardinality 2 for two values j, k such that $1 \leq j, k \leq p$ and order 1 for all other values of i. The subspace of such terms will be denoted $(A^{\#}[p,p+1] \otimes A^{\#}[p+1,p+2])'_{j,k}$.

Terms of the second type arise from the dual map to a sequence of two compositions, where the first composition can involve any pair from the three terms $A(n_j'), A(n_j''), A(n_j''')$ and the second composition involves the remaining term. These terms are indexed by pairs f, g such that $f^{-1}(g^{-1}(i))$ has order 3 for one value $i = j$ and order 1 for all other values of i. The subspace of such terms will be denoted $(A^{\#}[p,p+1] \otimes A^{\#}[p+1,p+2])''_j$.

Schematically we have the following picture before dualizing:



Type one: 4 new terms, two in position j and two in position k created in either order

Type two: 3 new terms, created in any cyclic permutation of n_j, n_j'', n_j'''

Looking at one component of $A^{\#}[p,n]$ we get the two types of terms in the image of $\tilde{\Delta}^2$:

$$\text{Type one :}\quad \bigotimes_{i=1}^{p} A^{\#}(n_i) \xrightarrow{\tilde{\Delta}^2} (A^{\#}[p,p+1]\otimes A^{\#}[p+1,p+2])'_{j,k}\otimes$$
$$\left(\bigoplus_{\substack{j,k=1,\cdots,p\\ n'_j+n''_j=n_j,n'_k+n''_k=n_k}} (A^{\#}(n'_j)\otimes A^{\#}(n''_j)\otimes A^{\#}(n'_k)\otimes A^{\#}(n''_k)\right)\otimes\left(\bigotimes_{i\neq j,k} A^{\#}(n_i))\right),$$

$$\text{Type two :}\quad \bigotimes_{i=1}^{p} A^{\#}(n_i) \xrightarrow{\tilde{\Delta}^2}$$
$$(A^{\#}[p,p+1]\otimes A^{\#}[p+1,p+2])''_j\otimes\left(\bigoplus_{\substack{j=1,\cdots,p\\ n'_j+n''_j+n'''_j=n_j}} (A^{\#}(n'_j)\otimes A^{\#}(n''_j)\otimes A^{\#}(n'''_j)\right)\otimes\left(\bigotimes_{i\neq j} A^{\#}(n_i)\right)$$
$$\cong (A^{\#}(T_1)\oplus A^{\#}(T_2)\oplus A^{\#}(T_3))\otimes\left(\bigoplus_{\substack{j=1,\cdots,p\\ n'_j+n''_j+n'''_j=n_j}} (A^{\#}(n'_j)\otimes A^{\#}(n''_j)\otimes A^{\#}(n'''_j)\right)\otimes\left(\bigotimes_{i\neq j} A^{\#}(n_i)\right).$$

The notation involving the binary trees T_i in the first factor of the last term is the same as in the proof of Theorem 3.2.14, and just as in that proof this factor is just $\Gamma(A^{\#}(2))(3)$. Modulo the tensor product with $\bigotimes_{i\neq j} A^{\#}(n_i)$, the component of type two is dual to the compositions described by

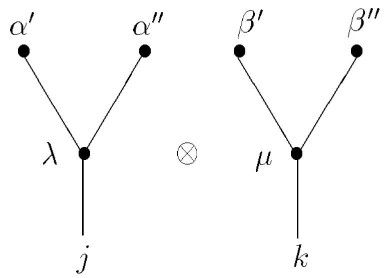
$$\bigoplus_{n'_j+n''_j+n'''_j=n_j} (A(T_1)\oplus A(T_2)\oplus A(T_3))\otimes\Big((A(n'_j)\otimes A(n''_j)\otimes A(n'''_j)\Big) \longrightarrow$$
$$\bigoplus_{n'_j+n''_j+n'''_j=n_j} A(2)\otimes\Big((A(n'_j+n''_j)\otimes A(n'''_j))\oplus(A(n'_j+n'''_j)\otimes A(n''_j))\oplus(A(n''_j+n'''_j)\otimes A(n'_j))\Big)$$
$$\longrightarrow A(n'_j+n''_j+n'''_j)=A(n_j).$$

By associativity, the composite arrow above is the same as the composite

$$\bigoplus_{n'_j+n''_j+n'''_j=n_j} (A(T_1)\oplus A(T_2)\oplus A(T_3))\otimes\Big(A(n'_j)\otimes A(n''_j)\otimes A(n'''_j)\Big) \longrightarrow$$
$$A(3)\otimes\Big(A(n'_j)\otimes A(n''_j)\otimes A(n'''_j)\Big)\longrightarrow A(n'_j+n''_j+n'''_j) \;=\; A(n_j).$$

Using the exact sequence (3.32) from the last part of the proof of Theorem 3.2.14 we see that the map dual to the compositions has image $R_A^{\perp}\otimes(A^{\#}(n'_j)\otimes A^{\#}(n''_j)\otimes A^{\#}(n'''_j))$. To calculate δ^2 we need to take the image of $R_A^{\perp}$ in $\Gamma(A^{\vee}(2))(3)$ and project to the operad $A^!$, but $R_A^{\perp}\to 0\in A^!(3)$, so this part of δ^2 is zero.

We can represent more precisely the component of type one in the image of $\tilde{\Delta}^2$ as a sum of terms:



The term containing $\lambda \otimes \mu$ pictured here occurs twice, once with $\lambda \in A^{\#}(2)$ created by the first application of $\tilde{\Delta}$ and then $\mu \in A^{\#}(2)$ created by the second application of $\tilde{\Delta}$, and once with μ created first and then λ. As a matter of notation we can assume that λ is to the left of μ so $j < k$. When λ is created first the term in $A^{\#}[p, p+1] \otimes A^{\#}[p+1, p+2] \otimes A^{\#}(n'_j) \otimes A^{\#}(n''_j) \otimes A^{\#}(n'_k) \otimes A^{\#}(n''_k)$ can be represented using the notation of Lemma 3.2.19 as

$$[\xi_{j,p} \otimes (1 \otimes \cdots \otimes \lambda) \otimes 1_{p+1}] \otimes [\xi_{k+1,p+1} \otimes (1 \otimes \cdots \otimes \mu) \otimes 1_{p+2}] \otimes (\alpha' \otimes \alpha'') \otimes (\beta' \otimes \beta'').$$

The first bracketed term is in $A^{\#}[p, p+1]$ with $\xi_{j,p} = (j(j+1) \cdots p) \in \Sigma_p$ and $1_{p+1} \in \Sigma_{p+1}$, the identity element. The second bracketed term is in $A^{\#}[p+1, p+2]$, with $\xi_{k+1,p+1} = ((k+1)(k+2) \cdots (p+1)) \in \Sigma_{p+1}$, and $1_{p+2} \in \Sigma_{p+2}$. When λ is created by the second application of $\tilde{\Delta}$, the corresponding term has the expression (since $j < k$)

$$[\xi_{k,p} \otimes (1 \otimes \cdots \otimes \mu) \otimes 1_{p+1}] \otimes [\xi_{j,p+1} \otimes (1 \otimes \cdots \otimes \lambda) \otimes 1_{p+2}] \otimes (\alpha' \otimes \alpha'') \otimes \beta' \otimes \beta'').$$

The definition of δ^2 involves composing $\tilde{\Delta}^2$ with the isomorphisms

$$\begin{aligned} \mathrm{sgn}_p \otimes A^{\#}[p, p+1] \otimes A^{\#}[p+1, p+2] &\cong A^{!}[p, p+1] \otimes \mathrm{sgn}_{p+1} \otimes A^{\#}[p+1, p+2] \\ &\cong A^{!}[p, p+1] \otimes A^{!}[p+1, p+2] \otimes \mathrm{sgn}_{p+2}. \end{aligned}$$

In the first case the isomorphism defined in Lemma 3.2.19 introduces a sign equal to

$$\mathrm{sgn}_p(\xi_{j,p})\mathrm{sgn}_{p+1}(\xi_{k+1,p+1}) = (-1)^{j+k},$$

whereas, in the second case the sign is $(-1)^{j+k+1}$; therefore, the two expressions coming from the two different orders of introducing the terms λ and μ have different signs and the two terms cancel. This completes the proof. ■

3.3. Free operad as a free monoid.

To prove Theorem 3.2.18 relating the definition of the Koszul property, Definition 3.2.16 and the Koszul complex $\mathbf{K}(A)$ we introduce yet anothe complex $\mathbf{N}(A)$ whose graded components are essentially iterated $\square_K$-products. The complex $\mathbf{N}(A)$ is related the description of the free operad as the free monoid for the $\square_K$-product, that is, a left-adjoint to the forgetful functor $\mathtt{Mon}_K \to \mathtt{Col}_K$. To simplify notation in the following discussion we usen $\square$ to stand for $\square_K$, $\mathbb{1}$ to stand for $\mathbb{1}_K$, and Γ to stand for Γ_K. This construction will be useful in our discussion of various types of Koszul resolutions.

Since $\square$ is not distributive over $\coprod$ on the right, the coproduct of iterated powers $\coprod A^{\square n}$ (Michael Barr's construction) is not closed under $\square$ product and therefore does not define a monoid, let alone a free monoid. However we can define a free monoid functor using a slight modification of the construction of Baues et. al. [3] to the case of K-collections.

Let A be a K-collection, $A(1) = K$, and define

$$\overline{A}(n) := \begin{cases} \mathbf{0}, & \text{for } n = 1, \text{ and} \\ A(n), & \text{for } n \geq 2. \end{cases}$$

Define a functor

$$V : \mathtt{Fun}(\widehat{\mathcal{S}}^{op}, \mathcal{C}) \to \mathtt{Fun}(\widehat{\mathcal{S}}^{op}, \mathcal{C}), \quad V(B) = \mathbb{1} \coprod (\overline{A} \square B), \quad V(f) = id_{\mathbb{1}} \oplus (id_A \square f). \tag{3.33}$$

Next define a sequence of $\widehat{\mathcal{S}}$-objects and morphisms using V, as follows.

$$A^0 = \mathbb{1}, \text{ and } A^{n+1} = V(A^n) \text{ for } n \geq 0;$$

$$i^1 : \mathbb{1} \to \mathbb{1} \coprod (\overline{A} \square \mathbb{1}), \; i^{n+1} = V(i^n) \text{ for } n > 1,$$

where i^1 is the inclusion of the first factor into the coproduct.

Define $M(A)$, the free monoid generated by the K-collection, A, to be the colimit of the diagram:

$$\mathbb{1} \xrightarrow{i^1} A^1 \xrightarrow{i^2} A^2 \xrightarrow{i^3} A^3 \xrightarrow{i^4} \cdots ,$$

with $\pi^p : A^p \to M(A)$ denoting the maps into the colimit. Define $M^p(A) := \pi^p(A^p)$, thereby producing an increasing filtration of $M(A)$. The fundamental difference between this construction and that of Barr is that the $M(A)$ is not graded but filtered.

The proof that $M(A)$ is a monoid depends on the easily verified fact that $\square$ is distributive over coproducts on the left and commutes with filtered colimits on the right. For a fixed m, define $\chi^{0,m} : A^0 \square A^m \cong \mathbb{1} \square A^m \to A^m$ to be the canonical isomorphism for the unit in $\mathtt{Col}_K$, and then $\chi^{p,m}$ is defined inductively in p to be the composition:

$$A^p \square A^m = (\mathbb{1} \coprod \overline{A} \square A^{p-1}) \square A^m \xrightarrow{\sim} (\mathbb{1} \square A^m) \coprod \left((\overline{A} \square A^{p-1}) \square A^m \right) \xrightarrow{\left(i, j[id \square \chi^{p-1,m}]a\right)} A^{p+m}.$$

The first arrow is left distributivity over coproducts, i is the composition $\mathbb{1}\Box A^m \to A^m \to A^{p+m}$, a is the associativity constraint for $\Box$, $\chi^{p-1,m}$ is given by the induction assumption, and j is the inclusion $\overline{A}\Box A^{p+m-1} \hookrightarrow A^{p+m}$. These maps pass to the colimit to produce a morphism $M(A)\Box M(A) \to M(A)$. The unit morphism $\mathbb{1} \to M(A)$ is clear. It is also clear that for all $A \in \mathtt{Mon}_K$ the morphism $A\Box A \to A$ and its iterates induces a morphism $M\mathcal{F}_{or}(A) \to A$ which gives a natural transformation $M\mathcal{F}_{or} \to id$. The isomorphism $A \cong A^1$ defines a morphism $A \to \mathcal{F}_{or}M(A)$ and a natural tranformation $id \to \mathcal{F}_{or}M$. Then standard arguments imply that M is left adjoint to the forgetful functor. The complete proof that this defines a free monoid functor can be found in [3, Appendix B, Proposition B.1]. Thus

$$M(A) = \mathcal{F}^{\Box}_{ree}(A) \text{ for } A \in \mathtt{Fun}(\widehat{\mathcal{S}}^{op}, \mathcal{C}). \tag{3.34}$$

In order to describe the relationship between $\Gamma(A)$ and $M(A)$ we consider the associated graded object of the filtered object $M(A)$. Consider first the initial stages of the recursive construction of $M(A)$:

$$A^0 = \mathbb{1}, \quad A^1 = \mathbb{1} \coprod \overline{A}\Box\mathbb{1} \cong \mathbb{1} \coprod \overline{A} \cong A, \quad A^2 = \mathbb{1} \coprod \overline{A}\Box A^1 \cong \mathbb{1} \coprod \overline{A}\Box A.$$

It will helpful in describing $(\overline{A}\Box A)(n)$ to use the following equivalence relation on the set of surjections $Surj[j,n]$:

$$f \sim g \text{ if and only if there exists } \sigma \in \Sigma_j,\ \sigma \circ f = g.$$

Let $[f]$ stand for the equivalence class of f and $\widetilde{Surj}[j,n]$ be the set of equivalence classes (which can be identified with the set of unordered partitions of n into j sets). Defining

$$A[[f]] := \coprod_{g\in[f]} A[g], \tag{3.35}$$

we can represent $(\overline{A}\Box A)(n)$ as the coproduct:

$$\coprod_{[f]\in\widetilde{Surj}[j,n],\, j\geq 2} A(j) \odot_{K[j,j]} A[[f]].$$

The morphism $i^1 : A^0 \to A^1$ is the unit morphism $\mathbb{1} \to A$ and the morphism $i^2 : A^1 \to A^2$ is

$$(\mathbb{1} \coprod \overline{A}\Box\mathbb{1}) \xrightarrow{id \coprod (id\Box i^1)} \mathbb{1} \coprod \overline{A}\Box A.$$

The components of $\overline{A}\Box A$ which are not in $\overline{A}\Box\mathbb{1}$ consist of terms $A(j) \odot_{K[j,j]} A[[f]]$ in which at least one factor $A(f^{-1}(i))$ in $A[[f]]$ is not equal to $\mathbb{1}$. Therefore a complement $L^2(A)$ to $\overline{A}\Box\mathbb{1}$ in $\overline{A}\Box A$ is given by

$$L^2(A)(n) := \coprod_{\substack{[f]\in\widetilde{Surj}[j,n],\, j\geq 2, \\ card(f^{-1}(i))\geq 2 \text{ for at least one } i}} A(j) \odot_{K[j,j]} A[[f]]. \tag{3.36}$$

At the second stage we have:

$$M^2(A) \cong M^1(A) \coprod L^2(A).$$

There is a generalization of this decomposition to the higher levels of the filtration of $M(A)$ given by defining $L^p(A)$ inductively so that

$$M^p(A) \cong M^{p-1}(A) \coprod L^p(A). \tag{3.37}$$

The definition of $M^p(A)$ as the image of A^p in $M(A)$ implies the isomorphism

$$M^{p+1}(A) \cong \mathbb{1} \coprod (\overline{A} \Box M^p(A)).$$

Defining $M^p(A)[[f]]$ by analogy to equation (3.35), we see that

$$\overline{A} \Box M^p(A) = \coprod_{[f] \in \widetilde{Surj}[j,n],\, j \geq 2} A(j) \odot_{K[j,j]} M^p(A)[[f]].$$

If the decomposition in equation (3.37) is known at level p then the complement $L^{p+1}(A)$ to $M^p(A)$ in $M^{p+1}(A)$ is given by the terms in $\overline{A} \Box M^p(A)$ which are not in $\overline{A} \Box M^{p-1}(A)$. Just as before these are the terms having at least one factor of $L^p(f^{-1}(i))$ appearing in $M^p(A)[[f]]$. Therefore if we define

$$\begin{aligned}(L/M)^{p+1}[f] := \coprod_{1 \leq s \leq j} M^p(f^{-1}(1)) \odot \quad \cdots \quad & \odot M^p(f^{-1}(s-1)) \odot L^p(f^{-1}(s) \odot \\ & M^p(f^{-1}(s+1)) \odot \cdots \odot M^p(f^{-1}(j))\end{aligned} \tag{3.38}$$

and

$$(L/M)^{p+1}[[f]] := \coprod_{g \in [f]} (L/M)^{p+1}[g], \tag{3.39}$$

then the complement to $M^p(A)$ in $M^{p+1}(A)$ is given by

$$L^{p+1}(A)(n) := \coprod_{[f] \in \widetilde{Surj}[j,n],\, j \geq 2} A(j) \odot_{K[j,j]} (L/M)^{p+1}[[f]]. \tag{3.40}$$

Coming back to the comparison to $\Gamma(A)$, we want to describe $L^p(A)$ in terms of trees. First we describe $L^2(A)$ by constructing the tree corresponding to a surjection.

Define a *strict surjection* to be a surjection which is not a bijection, that is $card(f^{-1}(i)) \geq 2$ for at least one i. Let $t(X)$ be the corolla with X as its set of leaves. When X is a singleton set, define $t(X)$ to be the trivial tree.

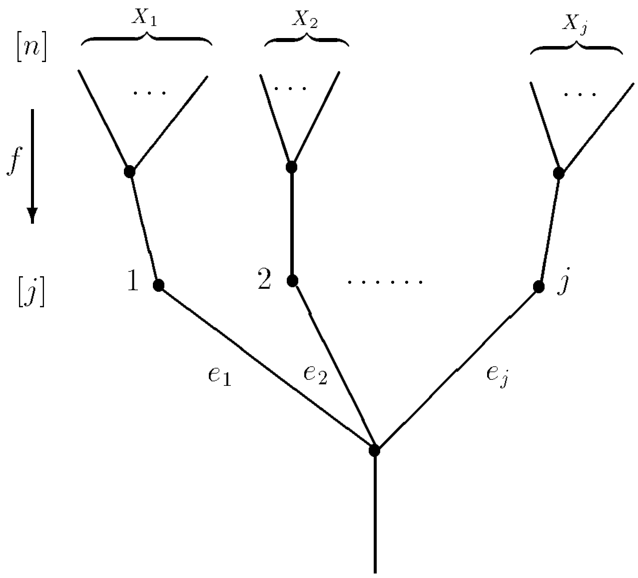


Figure 3.3: Constructing the tree T_f for $f : [n] \twoheadrightarrow [j]$.

Definition 3.3.1. *(The tree T_f corresponding to a surjection f) Given a strict surjection $f : [n] \twoheadrightarrow [j]$ such that $f^{-1}(i) = X_i$, define a tree T_f with j internal edges by grafting $t(X_i)$, onto $t([j])$ along the i-th leaf e_i,*

$$T_f = (\cdots(t([j]) \star_{e_1} t(X_1)) \star_{e_2} \cdots) \star_{e_j} t(X_j),$$

see Figure 3.3.

Note that the isomorphism class of the unlabeled tree T_f depends only on the equivalence class $[f]$ and it is clear from Definition 1.45 that

$$A(j) \odot_{K[j,j]} A[[f]] \cong A(T_f), \tag{3.41}$$

Using this definition we can identify $L^2(A)(n)$ described in (3.36) with a certain component of $\Gamma(A)$:

$$L^2(A)(n) \cong \coprod_{[f] \in \widetilde{Surj}[j,n],\ j \geq 2} A(T_f). \tag{3.42}$$

Assume that we have identified each $L^q(A)$ for $q \leq p$ with the coproducts of $A(T)$ for a certain class of trees depending on q. Then we can use the definitions (3.40) and (3.38) and the interpretation of the operation $\overline{A}\square(-)$ in terms of grafting to describe $L^{p+1}(A)$ as a coproduct of $A(T)$ for a new class of trees.

To describe the class of trees associated to $L^q(A)$ we need another definition.

Definition 3.3.2. *The* orientation partial ordering $\leq_o$ *on the vertices of a tree* T *is defined by*

$$v \leq_o v' \text{ if and only if } v \text{ lies on the (unique) path from } v' \text{ to the root vertex of } T.$$

A tree is said to have q *levels if the maximal ordered seqence of distinct vertices* $v_1 \leq_o v_2 \leq_o \cdots \leq_o v_r$ *has length* q. *Let* $\mathtt{Tree}^{red}(q)$ *denote the subcategory of* $\mathtt{Tree}^{red}$ *whose objects are trees with* q *levels, and* $\mathtt{Tree}^{red}_s(q)$, *trees with* q *levels and* s *leaves.*

Proposition 3.3.3. *There is a natural isomorphism of endofunctors on* $\mathtt{Fun}(\mathcal{S}^{op}, \mathcal{C})$

$$L^q(A) \longrightarrow \coprod_{T \in \mathtt{Tree}^{red}(q)} A(T)$$

which extends to a natural isomorphism of endofunctors on $\mathtt{Fun}(\mathcal{S}^{op}, \mathcal{C})$

$$M(A) \longrightarrow \Gamma(A).$$

Proof. There is an increasing filtration on $\mathtt{Tree}^{red}$ given by

$$F^r(\mathtt{Tree}^{red}) := \cup_{q \leq r} \mathtt{Tree}^{red}(q).$$

This induces a filtration on $\Gamma(A)$ with graded quotients $\coprod_{T \in \mathtt{Tree}^{red}(q)} A(T)$. To prove the second assertion of the proposition it is sufficient to prove the first assertion, since a morphism of filtered objects is an isomorphism if and only if the associated morphism of graded objects is an isomorphism. We have already proved the existence of an isomorphim for $q = 1, 2$, the naturality with respect to A is obvious. Suppose that the proposition has been proved for all $q \leq p$ and we will prove it for $p+1$. From the decomposition (3.37) it follows that

$$M^p(A) \cong A \coprod L^2(A) \coprod \cdots \coprod L^p(A).$$

Substituting this expression for $M^p(A)$ in (3.38) for $f \in Surj[j,n]$ with $X_i = f^{-1}(i)$ we have

$$(L/M)^{p+1}(A)[[f]](n) = \coprod_{\substack{1 \leq r_i \leq p \text{ at least one } r_i = p \\ \sigma \in \Sigma_j}} L^{r_1}(A)(X_{\sigma^{-1}(i)}) \odot \cdots \odot L^{r_j}(A)(X_{\sigma^{-1}(j)}).$$

By assumption we have an isomorphism between each factor $L^{r_i}(A)(X_{\sigma^{-1}(i)})$ and the coproduct of $A(T)$ over the reduced trees with leaves labeled by $X_{\sigma^{-1}(i)}$ and r_i levels:

$$L^{r_i}(A)(X_{\sigma^{-1}(i)}) \cong \coprod_{T \in \mathtt{Tree}^{red}_{X_{\sigma^{-1}(i)}}(r_i)} A(T).$$

Then taking $\odot_{K[j,j]}$, we have

$$\overline{A}(j) \odot_{K[j,j]} (L/M)^{p+1}(A)[[f]](n)$$

corresponds to the coproduct of $A(T)$ for all T given by grafting j-tuples of trees $T_i \in \mathtt{Tree}^{red}_{X_i}(r_i)$ (with at least one $r_i = p$) to the corolla with j leaves. Carrying out this operation for all $[f] \in \widetilde{Surj}[j,n]$ and all $1 < j < n$ gives the coproduct $\coprod_{T \in \mathtt{Tree}^{red}(p+1)} A(T)$, proving the induction. ■

3.4. Equivalent characterizations of the Koszul property.

In this section we define the complex $\mathbf{N}(A)$ and construct two spectral sequences for the cohomology of $\mathbf{N}(A)$, one has an $\mathbf{E}_1$ term which is isomorphic to the cobar complex $\mathbf{C}(A)$ and the other has an $\mathbf{E}_1$ term isomorphic to the Koszul complex $\mathbf{K}(A)$.

The definition of $\mathbf{N}(A)$ is based on an analogy to the theory of $\mathbf{k}$ algebras where the graded components of the cobar complex are iterated $\otimes$-products. However since the monoidal structure defined by $\square$ involves a non-trivial associativity constraint, the p-fold iterated product is not well-defined until we fix a placement of parentheses. We could fix the expansion of the parentheses, say expanding out from the left, but it is easier to use directly the right side of the following isomorphism which is an extension of the isomorphism in Proposition 1.2.10:

$$(\cdots((A\square A)\cdots A)\square A)(n) \cong \bigoplus_{1\leq r_1\leq r_2\leq\cdots\leq r_{p-1}\leq n} A(r_1)\otimes_{K[r_1,r_1]}A[r_1,r_2]\otimes\cdots\otimes_{K[r_{p-1},r_{p-1}]}A[r_{p-1},n]. \tag{3.43}$$

In contrast to the weak inequalities in (3.43), the complex will be further normalized by requiring strict inequalities. The component in degree p and arity n is defined by

$$\mathbf{N}(A)(n)^p := A^{\#}(n) \qquad \text{for } p=1, \tag{3.44}$$

$$:= \bigoplus_{1<r_1<r_2<\cdots<r_{p-1}<n} A^{\#}[r_1]\otimes_{K^{\rm op}[r_1,r_1]} A^{\#}[r_1,r_2]\otimes_{K^{\rm op}[r_2,r_2]}\cdots\otimes_{K^{\rm op}[r_{p-1},r_{p-1}]} A^{\#}[r_{p-1},n] \text{ for } p>1.$$

The composition law $\gamma : A\square A\to A$ for an operad A determines a family of morphisms

$$\gamma_{r,s,t} : A[r,s]\otimes_{K[s,s]} A[s,t] \hookrightarrow (A\square A)[r,t] \longrightarrow A[r,t],$$

for any triple of positive integers $1\leq r<s<t$. Dualizing, we get

$$\delta_{r,s,t} : A^{\#}[r,t] \longrightarrow A^{\#}[r,s]\otimes_{K^{\rm op}[s,s]} A^{\#}[s,t]. \tag{3.45}$$

Definition 3.4.1. *The* categorical cobar complex *in arity n is defined by*

$$\mathbf{N}(A)(n) := \bigoplus_{1\leq p\leq n-1} \mathbf{N}(A)(n)^p.$$

It is a double complex with internal differential $d^{\#}$ *given by the dg-structure on the tensor products of the $A^{\#}(r)$ and the corresponding* internal degree. *The $\square$-degree or* box degree *is the number of $\square$-product factors and the* box differential

$$\delta_{\mathbf{N}}(n)^p : \mathbf{N}(A)(n)^p \to \mathbf{N}(A)(n)^{p+1}$$

is defined on the component

$$\mathbf{N}(A)(r_1,\ldots,n) := A^{\#}[r_1]\otimes_{K^{\rm op}[r_1,r_1]} A^{\#}[r_1,r_2]\otimes_{K^{\rm op}[r_2,r_2]}\cdots\otimes_{K^{\rm op}[r_{p-1},r_{p-1}]} A^{\#}[r_{p-1},n]$$

by

$$\delta_{\mathbf{N}}(n)^p|_{\mathbf{N}(A)(r_1,\dots,n)} := \sum_{\substack{1\le i\le p \\ r_{i-1}<s_i<r_i}} (-1)^{i-1}\delta_{r_{i-1},s_i,r_i},$$

where $r_0 := 1$ *and* $r_p := n$. *The condition* $\delta_{\mathbf{N}}(n)^{p+1}\circ\delta_{\mathbf{N}}(n)^p = 0$ *follows from associativity of the operad composition. The compatibility of* $\delta_{\mathbf{N}}$ *and* $d^{\#}$ *follows from the assumption that* $A \in dg\text{-}\mathtt{Op}_K$. *The total differential is*

$$\delta := \sum_p (\delta_{\mathbf{N}}(n)^p + (-1)^p d^{\#}).$$

Remark 3.4.2. In [7] Ginzburg and Kapranov introduce a category $Cat(A)$ (actually a PROP) associated to an operad A. The objects are the natural numbers and the "Hom-sets," are $Hom_{Cat(A)}(n,m) := A[m,n]$. The categorical cobar complex is a subcomplex of the simplicial cochain complex of the classifying space of this category. They use this construction and results from an earlier paper of Beilinson, Ginzburg and Schechtman on Koszul categories, [7], to compare alternative characterizations of a Koszul operad. We present here the purely operadic proof given in [20].

The $\mathbf{N}(A)$ are closely related to, but not the same as, the graded collection $L(A^{\#})$ associated to the free monoid $M(A^{\#}) = \mathcal{F}^{\square}_{ree}(A^{\#})$ as defined in Section3.3, equation (3.37). To understand the structure of $\mathbf{N}(A)$ and its relation to $L(A^{\#})$ and the cobar complex $\mathbf{C}(A)$ we consider first the component of $\square$-degree 2:

$$(A^{\#}\square A^{\#})(n) \supset \mathbf{N}(A)(n)^2 = \bigoplus_{1<r<n} A^{\#}(r)\otimes_{K^{\mathrm{op}}[r,r]} A^{\#}[r,n] \cong \bigoplus_{\substack{1<r<n \\ [f]\in \widetilde{Surj}[r,n]}} A^{\#}(r)\otimes_{K^{\mathrm{op}}[r,r]} A^{\#}[[f]], \tag{3.46}$$

where the direct sum on the right is indexed by equivalence classes of surjections $f : [n] \to \to [r]$. We saw in Section 1.40, Definition 3.3.1 and eq (3.41) that there is a tree T_f defined independently of the choice of representative $f \in [f]$ such that

$$A^{\#}(r)\otimes_{K^{\mathrm{op}}[r,r]} A^{\#}[[f]] \cong A^{\#}(T_f).$$

This isomorphism together with the isomorphism in (3.46) presents $\mathbf{N}(A)^2$ in terms of a direct sum of $A^{\#}(T)$ as in $\mathbf{C}(A)$ except with different indexing:

$$\mathbf{N}(A)(n)^2 \cong \bigoplus_{\substack{1<r<n \\ [\mathrm{f}]\in \widetilde{Surj}[r,n]}} A^{\#}(T_f). \tag{3.47}$$

There is no twisting by $\det(T_f)$ but otherwise the summands are the same as those in the definition of $\mathbf{C}(A)$. However, the $\square$-degree of all the summands is 2 and the tree-degree of the summand $A^{\#}(T_f)$ is the number of vertices in T_f which could be as high as $[\frac{n}{2}]+1$.

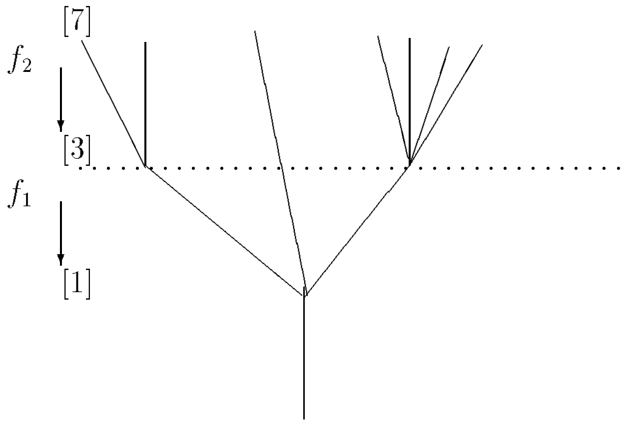


Figure 3.4: The tree representing the particular surjection sequence $[7] \overset{f_2}{\twoheadrightarrow} [3] \overset{f_1}{\twoheadrightarrow} [1]$ which indexes the component $A^{\#}(3) \otimes [A^{\#}(2) \otimes A^{\#}(1) \otimes A^{\#}(4)] \subset A^{\#}(3) \otimes A^{\#}[3,7] \subset \mathbf{N}^2(A^{\#})(7)$.

For $\Box$-degree $p > 2$, $\mathbf{N}(A)$ is not described as a direct sum indexed by trees as are $\mathbf{C}(A)$ and $L(A^{\#})$ (see Proposition 3.3.3). It is necessary to add an additional refinement in the indexing set, separating the vertices of a tree into levels. This refinement turns out to be the key to comparing the different forms of the Koszul condition. The direct sum in definition (3.44) can be refined into a direct sum indexed by equivalence classes of what we shall call surjection sequences, defined as follows.

Definition 3.4.3. *A* surjection sequence (surse) *of length p on the set $[n]$ is a sequence of p strict surjections (not bijections)*

$$[n] \overset{f_p}{\twoheadrightarrow} [r_{p-1}] \cdots \overset{f_2}{\twoheadrightarrow} [r_1] \overset{f_1}{\twoheadrightarrow} [1].$$

Such a surse is said to be of type $(r_1, \ldots, r_{p-1}, n)$. We reverse the order to fit with the order in the tensor product in (3.43) and denote a surse by $\mathbf{f} = (f_1, \ldots, f_p)$. Let SS_n^p be the set of all such sequences and $SS(r_1, \ldots, r_{p-1}, n)$ the subset of surse of type $(r_1, \ldots, r_{p-1}, n)$.

In the same way that a surjection $f : [n] \twoheadrightarrow [r]$ defines a partition of n into r parts, a surjection sequence defines a sequence of nested partitions.

We can describe $\mathbf{N}(A)$ as a direct sum over equivalence classes of strict surjection sequences, where the equivalence relation is defined by the action of the direct product of symmetric groups $\Sigma_{r_1} \times \cdots \times \Sigma_n$ on $SS(r_1, \ldots, r_{p-1}, n)$ given by the formula:

$$\hat{\sigma}\mathbf{f} = (\sigma_1, \ldots, \sigma_p)(f_1, \ldots, f_p) := (f_1, \sigma_1 \circ f_2 \circ \sigma_2^{-1}, \ldots, \sigma_{p-1} \circ f_p \circ \sigma_p^{-1}), \tag{3.48}$$

where $\hat{\sigma} = (\sigma_1, \ldots, \sigma_p)$ and $\mathbf{f} = (f_1, \ldots, f_p)$. The restricted action of $\Sigma_{r_1} \times \cdots \times \Sigma_{r_{p-1}}$ defines an equivalence relation on SS^p_n with the orbits as equivalence classes. Let $[\mathbf{f}]$ denote the equivalence class of $\mathbf{f}$, and $\widetilde{SS}^p_n$ the set of equivalence classes.

Remark 3.4.4. In defining the equivalence relation we do not use the full action of $\Sigma_{r_1} \times \cdots \times \Sigma_n$ but only of $\Sigma_{r_1} \times \cdots \times \Sigma_{r_{p-1}}$ so that there remains a residual action of Σ_n on the set of equivalence classes. A surjection sequence defines a sequence of nested ordered partitions of $[n]$. An equivalence class can be identified with a sequence of nested unordered partitions of $[n]$.

Definition 3.4.5. *For any surjection sequence* $\mathbf{f} \in SS(r_1, \ldots, n)$ *there are contravariant functors* $dg\text{-}\mathtt{Col}_K \longrightarrow dg\text{-}\mathtt{Mod}_{\mathbf{k}}$ *defined by*

$$\begin{aligned} A \mapsto A^{\#}[\mathbf{f}] &:= A^{\#}[f_1] \otimes_{(K^{\mathrm{op}})^{\otimes r_1}} A^{\#}[f_2] \otimes_{(K^{\mathrm{op}})^{\otimes r_2}} \cdots \otimes_{(K^{\mathrm{op}})^{\otimes r_{p-1}}} A^{\#}[f_p], \text{ and} \\ A \mapsto A[\mathbf{f}] &:= A[f_1] \otimes_{(K)^{\otimes r_1}} A[f_2] \otimes_{(K)^{\otimes r_2}} \cdots \otimes_{(K)^{\otimes r_{p-1}}} A[f_p]. \end{aligned}$$

For convenience in notation we define reduced forms $A^{\#}[\mathbf{f}]'$, *resp.* $A[\mathbf{f}]'$, *by deleting the factors* $A^{\#}[f_i^{-1}(j)]$, *resp.* $A[f_i^{-1}(j)]$ *for which* $|f_i^{-1}(j)| = 1$.

The next proposition brings in the symmetric group action which does not appear in the definition of $A[\mathbf{f}]$ or $A^{\#}[\mathbf{f}]$. We state the proposition only for $A^{\#}[\mathbf{f}]$ because that is the case we need, but obviously, there is a completely parallel proposition for $A[\mathbf{f}]$.

Proposition 3.4.6. *Given a* K*-collection* A*, there is an equivalence relation on*

$$\bigoplus_{\mathbf{f} \in SS(r_1, \ldots, n)} A^{\#}[\mathbf{f}] \text{ defined by}$$

$$A^{\#}[\mathbf{f}] \ni \alpha_1 \otimes \alpha_2 \otimes \cdots \otimes \alpha_p \sim \alpha_1\sigma_1 \otimes \sigma_1^{-1}\alpha_2\sigma_2 \otimes \cdots \otimes \sigma_{p-1}\alpha_p \in A^{\#}[\hat{\sigma}\mathbf{f}] \tag{3.49}$$

where $\hat{\sigma} = (\sigma_1, \ldots, \sigma_{p-1}) \in \Sigma_{r_1} \times \cdots \times \Sigma_{r_{p-1}}$. *The right action of* Σ_{r_i} *and the left action of* $\Sigma_{r_{i-1}}$ *on* $A[r_{i-1}, r_i]$ *were defined in equation (1.6) and Definition 1.2.4. Then*

$$\frac{\left(\bigoplus_{\mathbf{f} \in SS(r_1, \ldots, n)} A^{\#}[\mathbf{f}]\right)}{\sim} \cong A^{\#}[r_1] \otimes_{K[r_1, r_1]} A^{\#}[r_1, r_2] \otimes_{K^{\mathrm{op}}[r_2, r_2]} \cdots \otimes_{K^{\mathrm{op}}[r_{p-1}, r_{p-1}]} A^{\#}[r_{p-1}, n]. \tag{3.50}$$

Proof. The right side of (3.50) can be defined in two stages. First take the direct sum of $A^{\#}[\mathbf{f}]$ for $\mathbf{f} \in SS(r_1, \ldots, n)$. Since the $A[\mathbf{f}]$ are defined using the tensor products $\otimes_{K^{\otimes r_i}}$, the direct sum over all surse of the given type is

$$A^{\#}[r_1] \otimes_{(K^{\mathrm{op}})^{r_1}} A[r_1, r_2] \otimes_{(K^{\mathrm{op}})^{r_2}} \cdots \otimes_{(K^{\mathrm{op}})^{r_{p-1}}} A^{\#}[r_{p-1}, n].$$

Next take the quotient by the equivalence relation in order to introduce the symmetric group relations which appear in the definition of the tensor products $\otimes_{K^{\mathrm{op}}[r_i, r_i]}$:

$$\alpha_1 \otimes \cdots \otimes \alpha_i\sigma_i \otimes \alpha_{i+1} \otimes \cdots \otimes \alpha_p = \alpha_1 \otimes \cdots \otimes \alpha_i \otimes \sigma_i\alpha_{i+1} \otimes \cdots \otimes \alpha_p. \qquad \blacksquare$$

This allows us to define a functor $A^{\#}[[\mathbf{f}]]$ for each equivalence class $[\mathbf{f}]$.

Definition 3.4.7. *For any* $\mathbf{f} \in SS(r_1, \ldots, n)$ *the equivalence relation (3.49) restricts to* $\bigoplus_{\mathbf{f}\in[\mathbf{f}]} A^{\#}[\mathbf{f}]$ *and defines a functor* $dg\text{-}\mathtt{Col}_K \to dg\text{-}\mathtt{Mod}_{\mathbf{k}} : A \mapsto A^{\#}[[\mathbf{f}]] := (\bigoplus_{\mathbf{f}\in[\mathbf{f}]} A^{\#}[\mathbf{f}])/\sim$.

Using Proposition 3.4.6 and Definition 3.4.7 we can describe $\mathbf{N}(A)$ as a direct sum over equivalence classes of strict surjection sequences:

$$\mathbf{N}(A)(n)^p \cong \bigoplus_{[\mathbf{f}]\in\widetilde{SS^p_n}} A^{\#}[[\mathbf{f}]]. \tag{3.51}$$

If we allowed for one of the surjections in the sequence to be a bijection, we would have factors $A^{\#}[r, r]$. This is just like having tensor factor $\mathbf{k}$ in the cobar complex of the dual coalgebra of a finite dimensional $\mathbf{k}$-algebra. Removing this redundancy leads us to the normalized cochain complex $\mathbf{N}(A)(n)$ which is defined by strict inequalities.

Generalizing Definition 3.41, we will construct a tree $T_{\mathbf{f}}$ with leaves labeled by $[n] = \{1, \ldots, n\}$, associated to any equivalence class $[\mathbf{f}] \in \widetilde{SS^p_n}$ such that $\mathbf{N}(A)(n)^p$ for $p > 2$ can be presented as a direct sum of $A(T)$:

$$\mathbf{N}(A)(n)^p \cong \bigoplus_{[\mathbf{f}]\in\widetilde{SS^p_n}} A^{\#}(T_{\mathbf{f}}). \tag{3.52}$$

which once again looks like $\mathbf{C}(A)$ (without the twisting by $\det(T_{\mathbf{f}})$). As before the gradings look quite different; $\mathbf{N}(A)$ is graded by the length of the surjection sequence indexing a given summand and $\mathbf{C}(A)$ is graded by the number of vertices in the tree which indexes the summand. In fact, this isomorphism is a direct consequence of a natural isomorphism of functors

$$A^{\#}[[\mathbf{f}]] \xrightarrow{\cong} A^{\#}(T_{\mathbf{f}}) \tag{3.53}$$

given in Proposition 3.4.9 below.

Before describing the construction of $T_{\mathbf{f}}$ we remind the reader of some notation. For any finite set X with at least two elements, $t(X)$ is the corolla (rooted tree with one vertex) with leaves labeled by the set X. For $X = \{x\}$, the singleton set, $t(\{x\})$ is the trivial tree with one edge labeled by x and no vertices. As in the previous sections we consider only reduced trees for which there are no vertices with one incoming edge. In Definition 3.3.1 a surjection $f : [r] \twoheadrightarrow [s]$ was represented by an s-tuple of corollae $(t(X_1), \cdots, t(X_s))$ where $X_j = f^{-1}(j)$. In the present case we do this for each surjection in the surjection sequence, representing the surjection f_i by the ordered set of corollae $(t(X_{i,1}), \cdots, t(X_{i,s}))$ where $X_{i,j} = f_i^{-1}(j)$ and assigning this set to level i. Then we create a composite tree by grafting the corollae at level i to the leaves of the corollae at level $i-1$. The root vertex is considered as being at level 0, and there are no vertices corresponding to values onto which a surjection is one-to-one, which means a edge may pass from level j to level $j-1$ without introducing a vertex. The

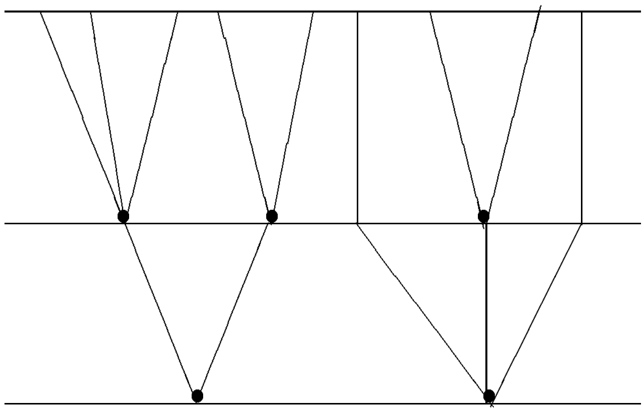


Figure 3.5: An example of the grafting of corollae at two adjacent levels of a surjection sequence. All vertices are indicated by a $\bullet$. Note that there are only three vertices created by the grafting.

vertices at level $j-1$ correspond to the many-to-one values of the surjection f_j. Since the surjection sequence does not contain any bijections, there is at least one vertex in each level.

More precisely, we have the following definition.

Definition 3.4.8. *Given a surse $\mathbf{f} = (f_1, \ldots, f_p)$, let $\mathbf{f}^j$ be the partial surse $(f_1, \ldots, f_{j-1})$. Define an "increasing sequence" of trees $T_{\mathbf{f}^{j-1}} \subset T_{\mathbf{f}^j}$, with $T_{\mathbf{f}^j}$ containing j levels of vertices (0 to $j-1$). The tree $T_{\mathbf{f}^j}$ is constructed by grafting, for each s such that $1 \leq s \leq r_{j-1}$ and $|f_j^{-1}(s)| > 1$, a corolla $t(f_j^{-1}(s))$ to the tree $T_{\mathbf{f}^{j-1}}$ along the leaf e_s. We define $T_{\mathbf{f}} := T_{\mathbf{f}_p}$. The vertices of $T_{\mathbf{f}^j} - T_{\mathbf{f}^{j-1}}$ are called either* the vertices at level $j-1$ *or* the vertices corresponding to f_j.

The construction is defined for a particular surjection sequence but as an abstract tree (with labeled leaves), $T_{\mathbf{f}}$ depends only on the equivalence class. Two equivalent surjection sequences differ only by relabeling the elements of the domain and range of the component surjections which amounts to relabeling the vertices, which is irrelevant to the description of $T_{\mathbf{f}}$ as an abstract tree.

Proposition 3.4.9. *The functors $A \mapsto A^{\#}[[\mathbf{f}]]$ and $A \mapsto A^{\#}(T_{\mathbf{f}})$ from dg-$\mathtt{Col}_K$ to dg-$\mathtt{Mod}_{\mathbf{k}}$ are naturally isomorphic.*

Proof. Any surse $\mathbf{f}$ defines an ordering of the vertices of $T_{\mathbf{f}}$. First, the vertices $\{v_1, \ldots, v_n\}$ are ordered by levels. Then the vertices at a given level are simply ordered by the values of the surjection defining that level, with the vertex corresponding to $f_j^{-1}(i)$ coming before the vertex corresponding to $f_j^{-1}(i')$, whenever $i < i'$. Then we have an isomorphism

$$A^{\#}[\mathbf{f}] = A^{\#}[f_1] \otimes A^{\#}[f_2] \otimes \cdots \otimes A^{\#}[f_n] \cong \bigotimes_{i=1}^{n} A^{\#}(In(v_i)), \tag{3.54}$$

where the tensor products are over K^{op} and the vertices are ordered as above. By definition, $A^{\#}(T_{\mathbf{f}})$ is the colimit of the $\bigotimes_{i=1}^{n} A^{\#}(In(v_i))$ with respect to the permutations of the vertices and for any particular ordering of the vertices

$$A^{\#}(T_{\mathbf{f}}) \cong \bigotimes_{i=1}^{n} A^{\#}(In(v_i)) \cong A^{\#}[\mathbf{f}].$$

These isomorphisms combine to give a morphism

$$\bigoplus_{\mathbf{f}\in[\mathbf{f}]} A^{\#}[\mathbf{f}] \longrightarrow A^{\#}(T_{\mathbf{f}}).$$

All the equivalence relations on $\bigoplus_{\mathbf{f}\in[\mathbf{f}]} A^{\#}[\mathbf{f}]$ defined by (3.49) are particular cases of the permutation relations defining the colimit $A(T_{\mathbf{f}})$. Hence there is an isomorphism

$$\psi_{[\mathbf{f}]} : A^{\#}[[\mathbf{f}]] \longrightarrow A^{\#}(T_{\mathbf{f}}), \tag{3.55}$$

and naturality of (3.55) relative to morphisms of dg-$\mathtt{Col}_K$ follows from the naturality of the isomorphism in (3.54). ■

We emphasize that if $\mathbf{N}(A)$ is described by the direct sum of $A(T_{\mathbf{f}})$ as in equation (3.52), then the $\square$-degree (ignoring internal degree from $A^{\#}$) is **not** the number of vertices but rather the number of levels of vertices.

Only the minimal cochain-degree components are isomorphic:

$$\mathbf{N}(A)(n)^1 = A^{\#}[1,n] \cong A^{\#}(n) = \mathbf{C}(A)(n)^1.$$

For the maximal cochain degree there is a surjection $\mathbf{N}(A)(n)^{n-1} \twoheadrightarrow \mathbf{C}(A)(n)^{n-1}$, which forgets the separation of vertices into levels:

$$\begin{aligned} \mathbf{N}(A)(n)^{n-1} \quad &= \quad A^{\#}[2] \otimes_{K^{\mathrm{op}}[2,2]} A^{\#}[2,3] \otimes_{K^{\mathrm{op}}[3,3]} \cdots \otimes_{K^{\mathrm{op}}[n-1,n-1]} A^{\#}[n-1,n] \\ &\longrightarrow \bigoplus_{\substack{T\in\mathtt{Tree}_n \\ |T|=n-2}} A^{\#}(T) \cong \mathbf{C}(A)(n)^{n-1}. \end{aligned}$$

The difference in the gradings of $\mathbf{C}(A)$ and $\mathbf{N}(A)$ makes the differentials quite different. The tree-differential in the cobar complex $\mathbf{C}(A)$ has 'matrix components' $\delta_{T,T'}$ one for each new tree T' created from the tree T by expanding a vertex into an internal edge. The box-differential in the categorical cobar complex $\mathbf{N}(A)$ creates one new level of vertices, and may create many new internal edges or no new internal edges. The latter possibility involves only the structure of A as a K-collection, and the resulting operation on the the reduced form simply separates the vertices at one level into two levels. For example, Figure 3.7 shows two ways of separating into levels two vertices adjacent to edges meeting in a common vertex. The notion of *homogeneity* of a surjection which we are about to define will allow us to keep track of the number of new vertices. We will use homogeneity to define a filtration of $\mathbf{N}(A)$ and an associated spectral sequence which relates the cohomologies of $\mathbf{N}(A)$ and $\mathbf{C}(A)$.

Definition 3.4.10. *The* homogeneity $\mathbf{h}(\mathbf{f})$ *of a surjection sequence* $\mathbf{f}$ *is the number of vertices in the tree* $T_{\mathbf{f}}$*:*

$$\mathbf{h}(\mathbf{f}) := |T_{\mathbf{f}}| + 1. \tag{3.56}$$

The homogeneity of an element $\alpha_1 \otimes \cdots \otimes \alpha_p \in A^{\#}[[\mathbf{f}]]$ *is the homogeneity of* $\mathbf{f}$*, which is the same as the number of* $\otimes$ *factors when the term is written in reduced form, c.f. (3.4.5).*

Alternatively we could define the homogeneity of a single surjection as the number of values for which f is many to one,

$$\mathbf{h}(f) := \#(\{j||f^{-1}(j)| > 1\}),$$

and the homogeneity of a surse $\mathbf{f} = (f_1, \ldots, f_p)$ as the sum of the homogeneities of its components

$$\mathbf{h}(\mathbf{f}) := \sum \mathbf{h}(f_i).$$

In the next definition we introduce the filtration that will be used to construct a spectral sequence whose $\mathbf{E}_1$-term is isomorphic to $\mathbf{C}(A)$.

Definition 3.4.11. *The* vertex filtration *is given by the subspaces* $F^s_v(\mathbf{N}(A)(n))$ *generated by all elements of homogeneity* $\geq s$*. This gives a decreasing filtration on* $\mathbf{N}(A)(n)$ *compatible with the differential.*

Separating the vertices of a tree into levels can be described in terms of order relations among the vertices. Given a tree T, any vertex is connected to the root vertex by a unique path. The vertices along this path are simply ordered with the root as minimal vertex. In this way we get a partial ordering on the vertices of the tree. Separating the vertices into levels refines this partial ordering.

Definition 3.4.12. *The T ordering on the vertices of a tree T is given by:*

$$v \leq_T v' \text{ if and only if } v \text{ lies on the path from } v' \text{ to the root.}$$

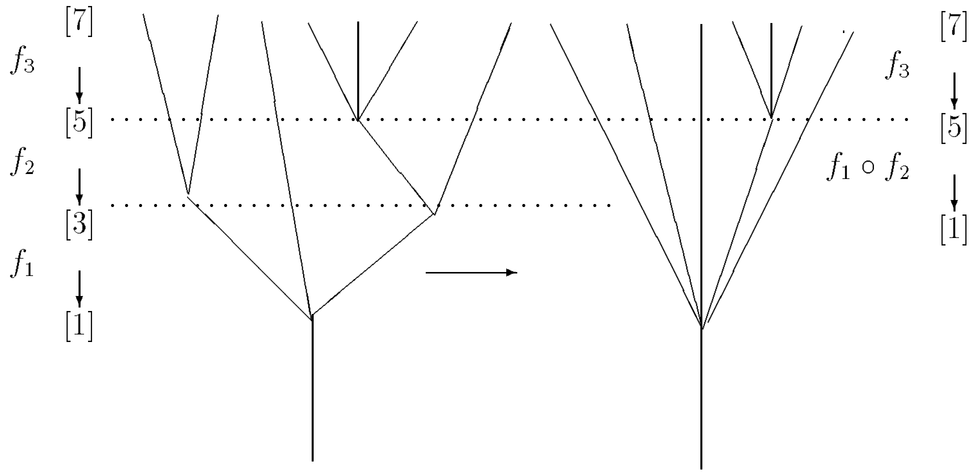


Figure 3.6: The three level tree for the surse $[7] \xrightarrow{f_3} [5] \xrightarrow{f_2} [3] \xrightarrow{f_1} [1]$ maps to the two level tree for the surse $[7] \xrightarrow{f_3} [5] \xrightarrow{f_1 \circ f_2} [1]$, collapsing two edges and decreasing homogeneity by 2. Reversing the arrow in this diagram describes one component of the $\square$-differential δ.

If $T = T_{\mathbf{f}}$ for surjection sequence $\mathbf{f}$ we can define another order on the vertices of T.

Definition 3.4.13. *Given a surse* $\mathbf{f} = (f_1, \ldots, f_p)$ *the* level function $\ell_{\mathbf{f}}$ *on the vertices of* $T = T_{\mathbf{f}}$ *is defined by:*

$\ell_{\mathbf{f}}(v) = j - 1$ *if and only if the vertex* v *corresponds to the a many to one value of the surjection* f_j. *Then the* $\mathbf{f}$-order *on the tree* $T_{\mathbf{f}}$ *is given by:*

$$v \leq_{\mathbf{f}} v' \text{ if and only if } \ell_{\mathbf{f}}(v) \leq \ell_{\mathbf{f}}(v').$$

Clearly the level function is independent of the choice of representative of the equivalence class $[\mathbf{f}]$. The next proposition follows immediately from the definitions.

Proposition 3.4.14. *The* $\mathbf{f}$-ordering *is a refinement of the* T*-ordering. The map* $[\mathbf{f}] \to T_{\mathbf{f}}$ *from* $\widetilde{SS}^p_n$ *(equivalence classes of surjection sequences) to trees with labeled leaves is many to one with inverse image of a tree* T *corresponding bijectively to the refinements of the orientation partial ordering of the vertices of* T *into a ordering into* p *levels.*

The ordering relates any two vertices but it is not a simple ordering since $v \leq_{\mathbf{f}} v'$ and $v' \leq_{\mathbf{f}} v$ does not imply $v = v'$.

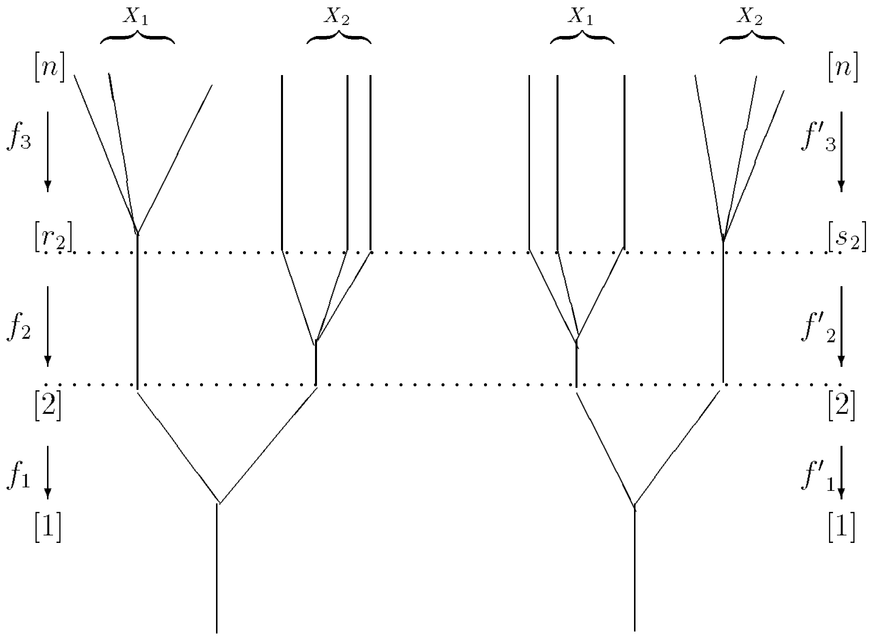


Figure 3.7: Two surjection sequences with the same tree, but different level functions. Each one determines a distinct component of the image of $\delta(A^{\#}[2] \otimes A^{\#}[2,r])$ as defined in Definition (3.4.1): $A^{\#}[2] \otimes A^{\#}[2,r_2] \otimes A^{\#}[r_2,r]$ on the left and $A^{\#}[2] \otimes A^{\#}[2,s_2] \otimes A^{\#}[s_2,r]$ on the right, where $r_2 = |X_2| + 1$ and $s_2 = |X_1| + 1$. Of course, there are many others as well.

Proposition 3.4.15. *There is a quasi-isomorphism betwee* $\mathbf{C}(A)$ *and* $\mathbf{N}(A)$ *coming from the spectral sequence for* $H^*(\mathbf{N}(A))$ *associated to the vertex filtration* F *defined in Definition 3.4.11.*

This proposition is stated at this point so that the reader will not get lost in the details of the rather technical discussion which follows. The proposition is actually a corollary of Theorem 3.4.21 below no proof is given here.

The term

$$\mathbf{E}_0^{p,q} := \mathbf{E}_0^{p,q}(\mathbf{N}(A)) := \frac{F^p(\mathbf{N}^{p+q}(A))}{F^{p+1}(\mathbf{N}^{p+q}(A)))}$$

in the initial stage of the spectral sequence, is spanned by reduced trees with exactly p vertices and $p+q$ levels, so we are dealing with "trees with levels", which also appear in the work of Loday, [13]. Since the number of levels is less than or equal to the number of vertices, $\mathbf{E}_0^{p,q} \neq 0$ if and only if $q \leq 0$.

The differential $\delta_0 : \mathbf{E}_0^{p,q} \longrightarrow \mathbf{E}_0^{p,q+1}$ preserves homogeneity, introducing one new level but no new vertices in $T_{\mathbf{f}}$. In order to compute the $\mathbf{E}_1$ term of the spectral sequence we need to describe δ_0 in more detail. The differential δ on $\mathbf{N}(A)$ (c.f., equation (3.45)) is the extension of $\delta_{r,s,t} : A^{\#}[r,t] \longrightarrow A^{\#}[r,s] \otimes_{K[s,s]} A^{\#}[s,t]$ defined by dualizing operad composition $A[r,s] \otimes_{K[s,s]} A[s,t] \longrightarrow A[r,t]$. For certain pairs of surjections and morphisms

$A[f] \otimes A[g] \to A[f \circ g]$ (those for which $\mathbf{h}(f \circ g) = \mathbf{h}(f) + \mathbf{h}(g)$) the only compositions are of the type $A(1) \otimes A(m) \longrightarrow A(m)$ or $A(m) \otimes A(1) \xrightarrow{\circ_i} A(m)$. Such compositions do not change homogeneity and involve only the structure of A as a K-collection, not the full operad structure. In such a composition, if the elements are expressed in reduced form, the only operation is a permutation of factors. Define a dual map to composition of this type

$$\delta'_{r,t} : A^{\#}[f] \longrightarrow \bigoplus_{\substack{f = f_1 \circ f_2 \\ h(f) = h(f_1) + h(f_2)}} A^{\#}[f_1] \otimes A^{\#}[f_2]. \tag{3.57}$$

Preserving homogeneity means that $[t] \overset{f}{\twoheadrightarrow} [r]$ and $[t] \overset{f_2}{\twoheadrightarrow} [s] \overset{f_1}{\twoheadrightarrow} [r]$ correspond to the same r-tuple of trees (see Figure 3.7) which implies that the factors $\alpha_i \in A^{\#}(f^{-1}(i))$ are partitioned into two subsets, but no new elements are created:

$$\delta'_{r,t}(\alpha_1 \otimes \cdots \otimes \alpha_r) = \sum_{\sigma \in \{(j, r-j)-\text{shuffles}\}} \epsilon(\sigma)(\alpha_{\sigma(1)} \otimes \cdots \otimes \alpha_{\sigma(j)}) \otimes (\alpha_{\sigma(j+1)} \otimes \cdots \otimes \alpha_{\sigma(r)}),$$

where $\epsilon(\sigma)$ is the Koszul sign factor (coming from the symmetry in dg-$\mathtt{Mod}_{\mathbf{k}}$). Therefore the differential $\delta_0 : \mathbf{E}_0^{p,q} \to \mathbf{E}_0^{p,q+1}$ acts on a component $A^{\#}[[\mathbf{f}]]$ mapping it to a sum of components $A^{\#}[[\bar{\mathbf{f}}]]$ where the trees $T_{\mathbf{f}}$ and $T_{\bar{\mathbf{f}}}$ are the same but the vertices of $T_{\mathbf{f}}$ are separated into $p+q$ levels and the vertices of $T_{\bar{\mathbf{f}}}$ are separated into $p+q+1$ levels. This can be interpreted as refining the $\mathbf{f}$-ordering of the vertices of the tree $T_{\mathbf{f}}$ into the $\bar{\mathbf{f}}$-ordering.

Proposition 3.4.16. *The following diagram commutes.*

$$\begin{tikzcd} A^{\#}[[\mathbf{f}]] \arrow[r, "\psi_{[\mathbf{f}]}"] \arrow[d, "\delta_0"'] & A^{\#}(T_{\mathbf{f}}) \arrow[d, "id"] \\ A^{\#}[[\bar{\mathbf{f}}]] \arrow[r, "\psi_{[\bar{\mathbf{f}}]}"] & A^{\#}(T_{\bar{\mathbf{f}}}), \end{tikzcd}$$

where $\psi_{[\mathbf{f}]}$ and $\psi_{[\bar{\mathbf{f}}]}$ are the isomorphisms defined in (3.55). In terms of the ordered tensor products $\bigotimes A^{\#}(In(v_i))$ defined in equation (3.54), the effect of δ_0 is a reordering of the tensor factors.

Proof. Before passing to equivalence classes, the matrix component $\delta_{0,\mathbf{f},\bar{\mathbf{f}}} : A^{\#}[\mathbf{f}] \longrightarrow A^{\#}[\bar{\mathbf{f}}]$ of the differential is just a reordering of the factors $A^{\#}(f_j^{-1}(i))$ of the reduced form. The permutations relating different surjection sequences in the equivalence class $[\bar{\mathbf{f}}]$ form a subset of the permutation relating elements of $[\mathbf{f}]$, with the restriction that they cannot permute values at different levels of $\bar{\mathbf{f}}$. Thus $\delta_{0,\mathbf{f},\bar{\mathbf{f}}}$ is induced by an inclusion

$$\iota : \bigoplus_{\bar{\mathbf{f}} \in [\bar{\mathbf{f}}]} A^{\#}[\bar{\mathbf{f}}] \hookrightarrow \bigoplus_{\mathbf{f} \in [\bar{\mathbf{f}}]} A^{\#}[\mathbf{f}],$$

and the composite isomorphism

$$\psi_{[\mathbf{f}]}\delta_{0,\mathbf{f},\bar{\mathbf{f}}} : A^{\#}[[\bar{\mathbf{f}}]] \longrightarrow A^{\#}[[\mathbf{f}]] \longrightarrow A^{\#}(T_{\mathbf{f}}) = A^{\#}(T_{\bar{\mathbf{f}}})$$

is the same as the isomorphism defined in (3.55) for $\bar{\mathbf{f}}$. ■

Let $(\mathbf{N}(A), \delta_{\mathbf{N}})$ be the categorical cobar complex with the $\square$-differential. In the spectral sequence relative to the vertex filtration the $\mathbf{E}_0^{p,*}$ term is a direct sum

$$\mathbf{E}_0^{p,*} \cong \bigoplus_{\{\mathbf{f}\,|\,\mathbf{h}(\mathbf{f})=p\}} A^{\#}[[\mathbf{f}]]. \tag{3.58}$$

For a tree T with p vertices, define

$$\mathbf{E}_0^{p,*}(T) := \bigoplus_{\{\mathbf{f}\,|\,T_{\mathbf{f}}=T\}} A^{\#}[[\mathbf{f}]]. \tag{3.59}$$

Then Proposition 3.4.16 implies that $\mathbf{E}_0^{p,*}(T)$ is a δ_0-subcomplex and we have the decomposition:

$$\mathbf{E}_0^{p,*} \cong \bigoplus_{\{T\,|\,|T|+1=p\}} \mathbf{E}_0^{p,*}(T) \tag{3.60}$$

In order to compute $\mathbf{E}_1$ we need to to understand the differential graded structure of $\mathbf{E}_0^{p,*}(T)$ For this purpose we introduce the following "surjection algebra".

Definition 3.4.17. *The* surjection algebra *B is the associative $\mathbf{k}$-algebra with basis $\{1\} \cup \{b_f\}$, where $\{b_f\}$ is indexed by the set of non-decreasing proper (not bijective) surjections of finite sets, that is the functions $\{h_{<m_1,\ldots,m_n>}\}$ from Lemma 1.2.14. The defining relations for multiplication of basis elements are*

$$b_f * b_g = \begin{cases} 0 & \text{if } f \circ g \text{ is not defined, (case 1)} \\ 0 & \text{if } f \circ g \text{ is defined, but } \mathbf{h}(f \circ g) < \mathbf{h}(f) + \mathbf{h}(g), \text{ (case 2)} \\ b_{f \circ g} & \text{if } f \circ g. \text{ is defined, and } \mathbf{h}(f \circ g) = \mathbf{h}(f) + \mathbf{h}(g) \text{ (case 3)} \end{cases}$$

If we use the same definition of homogeneity for the b_f as for the f, it is clear that the surjection algebra is generated as an algebra by elements b_f of homogeneity one. (In terms of Lemma 1.2.14, $f = h_{\langle 1,\ldots,1,i,1\ldots 1\rangle}$.) Moreover, the defining relations are quadratic:

$$b_f * b_g = \begin{cases} 0 & \text{case 1 or 2,} \\ b_{f'} * b_{g'} & \text{case 3 and } f \circ g = f' \circ g' \text{ as surjections,} \end{cases}$$

and therefore B is graded. The relations in case 3 simply say that all products of generators representing the same surjection are equal in the surjection algebra B.

We claim that B is a Koszul algebra in the classical sense defined by Priddy in his seminal paper [99]. The main point is that the special form of the relations insures the existence of a Poincaré-Birkhoff-Witt (PBW) basis for B.

Proposition 3.4.18. *The surjection algebra has a PBW basis.*

Proof. Recall the definition of a PBW basis from [99, Section 5.1] Such a basis consists of "admissible expressions" which are certain products of the generators (which have homogeneity one). Assume that a simple order has been defined on the generators. Then a lexicographic order is defined on the products of generators. Shorter words come first in the lexicographic ordering, and for two words of the same length, $w = b_1 * b_2 * \cdots * b_p$ and $w' = b'_1 * b'_2 * \cdots * b'_p$, w is smaller than w' if in the left-most position where the two words differ $b_i < b'_i$. The two conditions required for the admissible expressions are

1. If the product of admissible expressions $(b_1 * \cdots * b_p) * (b'_1 * \cdots * b'_q)$ is not itself an admissible expression, then the combined word is less than all the terms which appear in its representation as a linear combination of admissible expressions.
2. If $b_1 * b_2 * \cdots * b_p$ is an admissible expression then so are all the products $b_1 * b_2 * \cdots * b_q$ and $b_{q+1} * b_{q+2} * \cdots * b_p$ for $q < p$.

To show that the surjection algebra has a PBW basis, define a simple order on the (homogeneity one) generators. Then for any surjection g of homogeneity p, consider the set of all its factorizations into a product $b_g = b_{f_1} * \cdots * b_{f_p}$ of p generators of homogeneity one and pick as the the admissible expression representing b_g the word $b_{f_1} * b_{f_2} * \cdots * b_{f_p}$ which is maximal in the lexicographic ordering. From this definition it follows immediately that if the product of admissible expressions $\left(b_{f_1} * \cdots * b_{f_p}\right) * \left(b_{f'_1} * \cdots * b_{f'_p}\right)$ is not itself an admissible expression, then the admissible expression representing this product has a label which is greater. There is no need to consider linear combinations. This is condition (1) required of a PBW basis. Condition (2) is also an obvious consequence of our definition of "admissible expression" and the definition of lexicographical order ■

Corollary 3.4.19. *The surjection algebra B is a Koszul algebra. Therefore its cohomology algebra is isomorphic to the dual Koszul algebra $B^!$, which by definition has generators β_f corresponding bijectively to the generators b_f of B, and defining relations orthogonal to those of B:*

$\beta_f * \beta_g = -\beta_{f'} * \beta_{g'}$ *if the composition $f \circ g = f' \circ g'$ is defined and preserves homogeneity.*

Proof. See [99, Theorems 2.5 and 5.3]. ■

We now describe in more detail the structure of the cochain complex $C(B, \mathbf{k})$ for the (classical) Hochschild cohomology of the algebra B with coefficients in $\mathbf{k}$. Recall that $C^p(B, \mathbf{k})$

is the dual space $Hom_{\mathbf{k}}(B^{\otimes p}, \mathbf{k})$. The basis $\{b_{f_1} \otimes b_{f_2} \otimes ... \otimes b_{f_q}\}$ of $B^{\otimes q}$ splits into the union of two disjoint subsets

$$
\begin{aligned}
U_p &= \{b_{f_1} \otimes b_{f_2} \otimes ... \otimes b_{f_q} | \operatorname{domain}(f_{i-1}) \neq \operatorname{codomain}(f_i) \text{ for some } i\} \text{ and} \\
V_p &= \{b_{f_1} \otimes b_{f_2} \otimes ... \otimes b_{f_q} | \operatorname{domain}(f_{i-1}) = \operatorname{codomain}(f_i) \text{ for all } i\}.
\end{aligned}
$$

If $\overline{V_p} = \mathrm{Span}\,(V_p)$ and $\overline{U}_p = \mathrm{Span}\,(U_p)$, then $B^{\otimes p} = \overline{U}_p \oplus \overline{V_p}$. Letting $C_U^p = \mathrm{annihilator}(\overline{V}_p)$ and $C_V^p = \mathrm{annihilator}(\overline{U}_p)$, we have $C^p(B, \mathbf{k}) = C_U^p \oplus C_V^p$. The bar differential is defined by $\partial : B^{\otimes q} \to B^{\otimes (q-1)}$ by

$$
\partial(b_{f_1} \otimes ... \otimes b_{f_q}) = \sum (-1)^i b_{f_1} \otimes ... \otimes b_{f_i} * b_{f_{i+1}} \otimes ... \otimes b_{f_q},
$$

it follows that $\partial(\overline{U}_q) \subseteq \overline{U}_{q-1}$ and $\partial(\overline{V}_q) \subseteq \overline{V}_{q-1}$. Dualizing

$$
\delta C_U^q \subseteq C_U^{q+1}, \quad \delta C_V^q \subseteq C_V^{q+1}, \text{ and } C^*(B, \mathbf{k}) \cong C_U^* \oplus C_V^*. \tag{3.61}
$$

Let $\{b_f^\#\}$ be the dual basis to $\{b_f\}$; then C_V^q has a basis

$$
\{b_{\mathbf{f}} := b_{f_1}^\# \otimes b_{f_2}^\# \otimes \cdots \otimes b_{f_q}^\# | \operatorname{domain}(f_{i-1}) = \operatorname{codomain}(f_i), 1 < i \leq q\} \tag{3.62}
$$

and the differential is given by

$$
\begin{aligned}
\delta_B(b_{f_1}^\# \otimes b_{f_2}^\# \otimes ... \otimes b_{f_q}^\#) &= \sum (-1)^i b_{f_1}^\# \otimes ... \otimes \delta(b_{f_i}^\#) \otimes ... \otimes b_{f_q}^\#, \\
\delta_B(b_f^\#) &= \sum_{\{b_f = b_g * b_h\}} b_g^\# \otimes b_h^\#.
\end{aligned} \tag{3.63}
$$

There is a further direct sum decomposition

$$
C_V^* = \bigoplus C_V^*(T), \text{ where } C_V^q(T) = \mathrm{Span}\{b_{\mathbf{f}}^\# | T_{\mathbf{f}} = T\}. \tag{3.64}
$$

Proposition 3.4.20. *Let T be a tree in* $\mathtt{Tree}_n$ *and $p = (|T| + 1)$. The cohomologies of the subcomplexes $C_V^*(T)$ are as follows:*

$$
H^q(C_V(T)) = 0, \text{ for } 1 \leq q < p, \tag{3.65}
$$
$$
H^p(C_V(T)) \cong \mathbf{k}. \tag{3.66}
$$

Proof. The Koszul condition for B implies that the cohomology algebra is generated by $H^1(B)$ and therefore the homogeneity of any nonzero cohomology class equals its cochain degree. The direct sum decompositions (3.61) and (3.64) imply

$$
H(B, \mathbf{k}) \cong H(C_U^*) \oplus H(C_V^*), \text{ and } H(C_V) \cong \bigoplus_{T \in \mathtt{Tree}} H(C_V^*(T)).
$$

The homogeneity of any cochain $b^\#_{\mathbf{f}} = b^\#_{f_1} \otimes b^\#_{f_2} \otimes \ldots \otimes b^\#_{f_q} \in C^q_V(T)$ for the tree T in the statement of the proposition is p, the number of vertices in T. Therefore the cohomology $H(C^*_V(T))$ vanishes in dimensions $q < p$, proving the first assertion.

Corollary 3.4.19 implies that the p-th cohomology group of $C^*_V(T)$ is spanned by terms $\beta_{\mathbf{f}} := \beta_{f_1} * \beta_{f_2} * \cdots * \beta_{f_p}$ for surjection sequences $\mathbf{f}$ such that $T_{\mathbf{f}} = T$, subject to the relations

$$\beta_{f_1} * \cdots * \beta_{f_i} * \beta_{f_{i+1}} * \cdots \beta_{f_p} = -\beta_{f_1} * \cdots * \beta_{f'_i} * \beta_{f'_{i+1}} * \cdots \beta_{f_p} \tag{3.67}$$

for any two surjection sequences

$$\begin{aligned} \mathbf{f} &= [n] \xrightarrow{f_p} [r_{p-1}] \cdots [r_{i+1}] \xrightarrow{f_{i+1}} [r_i] \xrightarrow{f_i} [r_{i-1}] \cdots [r_1] \xrightarrow{f_1} [1], \text{ and} \\ \mathbf{f}' &= [n] \xrightarrow{f_p} [r_{p-1}] \cdots [r_{i+1}] \xrightarrow{f'_{i+1}} [r'_i] \xrightarrow{f'_i} [r_{i-1}] \cdots [r_1] \xrightarrow{f_1} [1], \end{aligned}$$

differing in two adjacent positions, but determining the same tree. See Figure 3.7.

We claim that this space is one dimensional. Let $o_{\mathbf{f}}$ be the ordering of the vertices of T coming from the surjection sequence $\mathbf{f}$, and define the set of $\mathbf{f}$-orderings of T:

$$o(T) := \{o_{\mathbf{f}} | \mathbf{f} \in SS^p_n,\ T_{\mathbf{f}} = T\} \tag{3.68}$$

Consider two orderings $o_{\mathbf{f}}$ and $o_{\mathbf{f}'}$ which are related by transposing the order of two vertices v, w in adjacent levels, that is the orders agree except on the vertices v, w but

$$v \leq_{\mathbf{f}} w, \text{ whereas } v \geq_{\mathbf{f}'} w.$$

Equation (3.67) then implies that the corresponding elements $\beta_{\mathbf{f}}$ and $\beta_{\mathbf{f}'}$ are equal up to a change of sign. Since any two orderings are related by a sequence of such transpositions, the space generated by the $\beta_{\mathbf{f}}$ is one dimensional and carries the signum representation of the subgroup of Σ_p (acting as permutations of the vertices of T) which stabilizes the set $o(T)$. ■

Theorem 3.4.21. *Let T be a tree with $|T| + 1 = p$.*

1. There is a dg-$\mathtt{Mod}_{\mathbf{k}}$ *isomorphism*

$$(\mathbf{E}^{p,*}_0(T), \delta_0) \cong (A^\#(T) \otimes C^*_V(T), id \otimes \delta_B). \tag{3.69}$$

2. At the next stage of the spectral sequence, the $\mathbf{E}^{p,*}_1(T)$ term is given by:

$$\begin{aligned} \mathbf{E}^{p,q}_1(T) &:= H^{p+q}(\mathbf{E}^{p,*}_0(T)) = \langle 0 \rangle \textit{ for } q < 0, \textit{ and} \\ \mathbf{E}^{p,0}_1(T) &:= H^p(\mathbf{E}^{p,0}_0(T)) \cong A^\#(T). \end{aligned} \tag{3.70}$$

Taking the direct sum over T one has:

$$\begin{aligned}\mathbf{E}_1^{p,q} &= \langle 0\rangle,\ q<0,\\ \mathbf{E}_1^{p,0} &\cong \bigoplus_{\{|T|=p-1\}} A^{\#}(T)\cong \mathbf{C}(A)^p.\end{aligned} \tag{3.71}$$

3. The isomorphisms in (3.71) combine to give an dg-$\mathtt{Mod}_{\mathbf{k}}$ *isomorphism*

$$\mathbf{E}_1^{*,0}\cong \mathbf{C}(A) \tag{3.72}$$

Proof. The isomorphism (3.69) follows from the definitions (3.59) and (3.64), Proposition 3.4.16 and the definition of the surjection algebra. The description of the $\mathbf{E}_1$ term as a graded $\mathbf{k}$-vector space follows immediately from Proposition 3.4.20.

The differential δ_1 increases the number of vertices in a tree by one, and is dual to the operad composition law for the operad E. This is precisely the definition of the differential δ on the cobar-dual (tree complex), $\mathbf{C}(A)$. ■

We now use these results to prove the equivalence of two characterizations of the Koszul condition for a quadratic operad. Recall that according to Definition 3.2.17, $K(A^!)(n)$ is called the Koszul complex of the operad A.

Theorem 3.4.22. *A quadratic operad A is Koszul if and only if the Koszul complex of A is exact for all $n\geq 2$.*

Proof. The proof is based on another spectral sequence coming from the following decreasing filtration of the complex $\mathbf{N}(A)$,

$$\begin{aligned}\text{For } p>1:\quad & F^s\Big(\bigoplus A^{\#}(r_1)\otimes_{K^{\mathrm{op}}[r_1,r_1]}\cdots\otimes_{K^{\mathrm{op}}[r_{p-1},r_{p-1}]}A^{\#}[r_{p-1},n]\Big)\\ &= \bigoplus_{n+s<r_{p-1}<n} A^{\#}(r_1)\otimes_{K^{\mathrm{op}}[r_1,r_1]}\cdots\otimes_{K^{\mathrm{op}}[r_{p-1},r_{p-1}]}A^{\#}[r_{p-1},n],\\ \text{For } p=1,\ s\leq -n+1:\ & F^s\Big(A^{\#}(n)\Big) = A^{\#}(n)\\ \text{For } p=1,\ s\geq -n+2:\ & F^s\Big(A^{\#}(n)\Big) = \langle 0\rangle.\end{aligned}$$

The filtration at arity n is strictly decreasing for $-n+1\leq s\leq 0$:

$$\mathbf{N}(A)(n)=F^{-n+1}(\mathbf{N}(A)(n))\supset\ldots\supset F^{-1}(\mathbf{N}(A)(n))\supset F^0(\mathbf{N}(A)(n))=\langle 0\rangle.$$

For $-n+2\leq s<0$ and $s+t>1$

$$\hat{\mathbf{E}}_0^{s,t}(\mathbf{N}(A)(n))=\frac{F^s(\mathbf{N}(A)(n)^{s+t})}{F^{s+1}(\mathbf{N}(A)(n)^{s+t})}\cong \mathbf{N}(A)[n+s]^{s+t-1}\otimes_{K^{\mathrm{op}}[n+s,n+s]}A^{\#}[n+s,n]. \tag{3.73}$$

For $s+t=1$,

$$\hat{\mathbf{E}}_0^{s,t}(\mathbf{N}(A)(n)) = A^{\#}(n), \text{for } t=n, \text{ and } \hat{\mathbf{E}}_0^{s,t}(\mathbf{N}(A)(n)) = \langle 0\rangle, \text{ for } t \neq n.$$

Let $\delta(n)$ denote the differential on $\mathbf{N}(A)(n)$ and denote all differentials associated to the spectral sequence accordingly, then the differential on $\hat{\mathbf{E}}_0^{s,*}(\mathbf{N}(A)(n))$ is given by $\delta_0^s(n) = \delta[n+s]\otimes 1$. Therefore, for $-n+2 \leq s < 0$ and $s+t>1$,

$$\hat{\mathbf{E}}_1^{s,t}(\mathbf{N}(A)(n)) = H^{s+t-1}(\mathbf{N}(A)(n+s)) \otimes_{K^{\mathrm{op}}[n+s,n+s]} A^{\#}[n+s,n].$$

To prove the theorem in one direction, assume that the operad E is Koszul, which according to Definition 3.2.16 means that

$$H^t(\mathbf{C}(A)(n)) = 0 \text{ for } t < n-1, \text{ and } H^{n-1}(\mathbf{C}(A)(n)) \cong A^!(n)\otimes \mathrm{sgn}_n.$$

The sgn_n factor appears because we are using $\mathbf{C}(A)$ instead of $\mathbf{D}(A)$, see Definition 3.1.12. By Theorem 3.4.21 this is equivalent to

$$H^t(\mathbf{N}(A)(n)) = 0 \text{ for } t < n-1, \text{ and } H^{n-1}(\mathbf{N}(A)(n)) \cong A^!(n)\otimes \mathrm{sgn}_n.$$

Therefore at the $\hat{\mathbf{E}}_1$ stage of the spectral sequence we have only one non-zero row, $t=n$:

$$\begin{aligned} \hat{\mathbf{E}}_1^{r-n,n}(\mathbf{N}(A)(n)) &\cong A^![r]\otimes_{K^{\mathrm{op}}[r,r]}\otimes \mathrm{sgn}_r \otimes A^{\#}[r,n],\ r>1 &(3.74)\\ \hat{\mathbf{E}}_1^{1-n,n}(\mathbf{N}(A)(n)) &\cong A^{\#}(n). &(3.75)\end{aligned}$$

This is just the Koszul complex truncated at the last term $A^!(n)$. The fact that $\hat{\mathbf{E}}_1$ is supported on a single row means that $\hat{\mathbf{E}}_2 \cong \hat{\mathbf{E}}_\infty$. Then

$$\begin{aligned} H^{n-1}(\hat{\mathbf{E}}_1^{*,n}) = \hat{\mathbf{E}}_2^{-1,n}(\mathbf{N}(A)(n)) &\cong A^!(n)\otimes \mathrm{sgn}_n \text{ and}\\ H^{t+n}(\hat{\mathbf{E}}_2^{*,n}) = \hat{\mathbf{E}}_2^{t,n}(\mathbf{N}(A)(n)) &= \langle 0\rangle \text{ for } t\neq -1.\end{aligned}$$

Thus the augmented complex $\hat{\mathbf{E}}_1^{*-n,n}(\mathbf{N}(A)(n) \longrightarrow A^!(n)\otimes \mathrm{sgn}_n$ is exact. But this is the full Koszul complex, which proves the theorem in one direction, "Koszul implies exactness of the Koszul complex".

To prove the theorem in the other direction we must prove the acyclicity of the augmented complex

$$\mathbf{N}(A)(n) \longrightarrow A^!(n)\otimes \mathrm{sgn}_n. \tag{3.76}$$

assuming that the Koszul complexes are exact. The proof is by induction on n. For $n=2$, $\mathbf{N}(A)(2)^1 = A^{\#}(2)$ and so the complex (3.76) expresses the isomorphism $A^{\#}(2) \cong A^!(2)\otimes \mathrm{sgn}_2$. For $n=3$ the complex (3.76)is the Koszul complex and so is exact by assumption. Suppose we have shown that this complex is exact for $r<n$, and we want to prove that it is exact for n. The same spectral sequence that we used in the first part of the proof shows that $\hat{\mathbf{E}}_1^{r-n,n}(\mathbf{N}(A)(n)) \cong A^![r]\otimes_{K^{\mathrm{op}}[r,r]} \mathrm{sgn}_r \otimes A^{\#}[r,n]$, since the only indices r which enter the argument are strictly less than n. Once again $\hat{\mathbf{E}}_1$ is supported on a single row, so $\hat{\mathbf{E}}_2 \cong \hat{\mathbf{E}}_\infty$ which is, as before, the Koszul complex'truncated by the last term $A^!(n)$. By assumption the Koszul complexes are exact so $\hat{\mathbf{E}}_\infty^{-1,n} \cong A^!(n)$ and therefore the augmented complex $\mathbf{N}(A)(n) \longrightarrow A^!(n)\otimes \mathrm{sgn}_n$ is acyclic. ■

3.5. Operad algebras and their cohomology.

Let $(\mathcal{C}, \odot)$ be a symmetric closed monoidal category with "internal Hom" functor, $\underline{Hom}_{\mathcal{C}}$. Recall the definition of the endomophism operad End_X for an object $X \in \mathcal{C}$

$$End_X(n) := \underline{Hom}_{\mathcal{C}}(X^{\odot n}, X). \tag{3.77}$$

Definition 3.5.1. *Let A be an operad in $\mathcal{C}$. An A-*algebra structure *on the object $X \in \mathcal{C}$ is a morphism of operads $\alpha_X : A \longrightarrow End_X$, that is, for $n \geq 1$ a family of Σ_n-equivariant morphisms called* structure morphisms

$$\alpha_X(n) : A(n) \longrightarrow End_X(n).$$

Equivalently, an A-algebra structure on X is a family of composition morphisms *in $\mathcal{C}$*

$$\hat{\alpha}_X(n) : A(n) \odot_{\Sigma_n} X^{\odot n} \longrightarrow X.$$

We also say that X is an algebra over *A.*

The equivalence of the two definitions follows immediately from the property of $\underline{Hom}_{\mathcal{C}}$

$$Hom_{\mathcal{C}}(A(n), \underline{Hom}_{\mathcal{C}}(X^{\odot n}, X)) \longleftrightarrow Hom_{\mathcal{C}}(A(n) \odot X^{\odot n}, X)$$

and the fact that requiring a structure morphism $\alpha_X \in Hom_{\mathcal{C}}(A(n), \underline{Hom}(X^{\odot n}, X))$ to be Σ_n equivariant is equivalent to requiring that the corresponding composition morphisms $\hat{\alpha}_X \in Hom_{\mathcal{C}}(A(n) \odot X^{\odot n}, X)$ factor through $A(n) \odot_{\Sigma_n} X^{\odot n}$.

Definition 3.5.2. *Given two A-algebras, X and Y, a morphism $\varphi \in Hom_{\mathcal{C}}(X, Y)$ is a morphism of A-algebras if the following diagram commutes for all $n \geq 1$:*

$$\begin{tikzcd}
A(n) \odot X^{\odot n} \arrow[r, "\hat{\alpha}_X(n)"] \arrow[d, "id \odot \varphi^{\odot n}"'] & X \arrow[d, "\varphi"] \\
A(n) \odot Y^{\odot n} \arrow[r, "\hat{\alpha}_Y(n)"] & Y
\end{tikzcd}.$$

Remark 3.5.3. The definition of A-algebra is formulated so that Ass-algebras, Lie-algebras, and Com-algebras for the category dg-$\mathtt{Mod}_{\mathbf{k}}$ coincide with the standard definitions.

Given an object $X \in \mathcal{C}$, define the graded object

$$\mathcal{F}_A(X) := \coprod_{n \geq 1} A(n) \odot_{\Sigma_n} X^{\odot n}. \tag{3.78}$$

Proposition 3.5.4. *The morphisms*

$$\begin{aligned} \alpha(n) &:= \alpha_{\mathcal{F}_A(X)}(n) : A(n) \odot \mathcal{F}_A(X)^{\odot n} \longrightarrow \mathcal{F}_A(X), \\ \alpha(n) &:= \coprod_{(m_1,\dots,m_n)} \alpha(n; m_1, \dots, m_n) \\ \alpha(n; m_1, \dots, m_n) &: A(n) \odot \Big((A(m_1) \odot_{\Sigma_{m_1}} X^{\odot m_1}) \odot \cdots \odot (A(m_n) \odot_{\Sigma_{m_n}} X^{\odot m_n}) \Big) \\ &\cong A(n) \odot (A(m_1) \odot \cdots \odot A(m_n)) \odot_{\Sigma_{m_1} \times \cdots \times \Sigma_{m_n}} \Big(X^{\odot m_1} \odot \cdots \odot X^{\odot m_n} \Big) \\ &\xrightarrow{\gamma(n; m_1, \cdots, m_n) \odot id} A(m_1 + \cdots + m_n) \odot_{\Sigma_{m_1 + \cdots + m_n}} X^{\odot (m_1 + \cdots + m_n)} \end{aligned}$$

define an A-algebra structure on $\mathcal{F}_A(X)$, with respect to which it is a free A-algebra.

Proof. The composition morphisms for $\mathcal{F}_A(X)$ are induced by the symmetries of the symmetric monoidal category and the composition morphisms for the operad A, therefore the equivariance axioms for $\mathcal{F}_A(X)$ follow from the naturality of the symmetries and the equivariance axioms for the operad A.

The condition for $\mathcal{F}_A(X)$ to be a free A-algebra is the existence two natural transformations of functors (adjunctions)

$$\zeta : Id \longrightarrow U\mathcal{F}_A, \text{ and } \chi : \mathcal{F}_A U \longrightarrow Id,$$

where U is the forgetful functor from the category of A-algebras to $\mathcal{C}$, establishing a bijective correspondence

$$\begin{aligned} Hom_{A-alg}(\mathcal{F}_A(X), Y) &\longleftrightarrow Hom_{\mathcal{C}}(X, U(Y)), \\ \varphi &\longmapsto U(\varphi) \circ \zeta_X \\ \chi_Y \circ \mathcal{F}_A(\psi) &\longleftarrow\!\!| \ \psi \end{aligned}$$

Assuming as usual $A(1) \cong \mathbb{1}$, the unit object in $\mathcal{C}$, we define $\zeta_X \in Hom_{\mathcal{C}}(X, U\mathcal{F}_A(X))$, as the composition of the sequence of morphisms

$$X \cong \mathbb{1} \odot X \cong A(1) \odot X \longrightarrow \coprod A(n) \odot_{\Sigma_n} X^{\odot n} = \mathcal{F}_A(X).$$

On the other hand, if X is an A-algebra the structure, then morphisms $A(n) \odot X^{\odot n} \longrightarrow X$ determine a morphism of the coproduct

$$\chi_X : \coprod A(n) \odot_{\Sigma_n} X^{\odot n} \longrightarrow X.$$

This defines the adjunctions.

Bijectivity of the correspondence follows immediately from the commutative diagram for an A-algebra morphism:

$$\begin{tikzcd} A(n)\odot(A(1)\odot X)^{\odot n} \arrow[rr, "\alpha_{\mathcal{F}_A(X)}(n)"] \arrow[d, "id\odot\varphi_1^{\odot n}"'] & & A(n)\odot X^{\odot n} \arrow[d, "\varphi_n"] \\ A(n)\odot Y^{\odot n} \arrow[rr, "\alpha_Y(n)"] & & Y \end{tikzcd}$$

which shows that $\varphi_1 := U(\varphi)\circ\zeta_X$ determines uniquely all the higher φ_n. ■

Next we introduce a homology theory for algebras over quadratic operads in the category dg-$\mathtt{Mod}_{\mathbf{k}}$, which coincides with the standard theories, Hochschild, Harrison, Chevalley-Eilenberg for *Ass*, *Com* and *Lie* respectively, and leads to a very useful characterization of the Koszul property: a quadratic operad A is Koszul if and only if the homology of $\mathcal{F}_A(X)$ vanishes in all positive degrees and equals X in degree 0.

Definition 3.5.5. *Let A be a quadratic operad in dg-$\mathtt{Mod}_{\mathbf{k}}$ and Y an A-algebra. Define the* A-algebra homology complex *of Y:*

$$C^A_{n-1}(Y) := (A^!(n)^\# \otimes \mathrm{sgn}_n)\otimes_{K[n,n]} Y^{\otimes n} \text{ for } n>1.$$

The differential is given by the composite morphism

$$\begin{aligned} \partial^A : (A^!(n)^\# \otimes \mathrm{sgn}_n)\otimes_{K[n,n]} \quad & Y^{\otimes n} \\ \longrightarrow \quad & (A^!(n-1)^\# \otimes_{K[n-1,n-1]} (A^![n-1,n]^\# \otimes \mathrm{sgn}_n))\otimes_{K[n,n]} Y^{\otimes n} \\ \longrightarrow \quad & (A^!(n-1)^\# \otimes_{K[n-1,n-1]} (\mathrm{sgn}_{n-1}\otimes A[n-1,n]))\otimes_{K[n,n]} Y^{\otimes n} \\ \cong \quad & (A^!(n-1)^\# \otimes \mathrm{sgn}_{n-1})\otimes_{K[n-1,n-1]} A[n-1,n]\otimes_{K[n,n]} Y^{\otimes n} \\ \longrightarrow \quad & (A^!(n-1)^\# \otimes \mathrm{sgn}_{n-1})\otimes_{K[n-1,n-1]} Y^{\otimes n-1} \end{aligned}$$

The first arrow is dual to the $A^!$ composition morphisms :

$$A^!(n-1)\otimes_{K^{\mathrm{op}}[n-1,n-1]} A^![n-1,n] \longrightarrow A^!(n).$$

The second arrow is the isomorphism from Lemma 3.2.19:

$$A^![n-1,n]^\# \otimes \mathrm{sgn}_n \cong \mathrm{sgn}_{n-1}\otimes (A^!)^\vee[n-1,n] \cong \mathrm{sgn}_{n-1}\otimes A[n-1,n].$$

The last arrow is the extension of the structure morphism $A(2)\otimes_{K[2,2]} Y^{\otimes 2}\longrightarrow Y$.

Remark 3.5.6. By the universal properties of free algebras, in order to show that ∂^A as defined in (3.79) has square zero it is sufficient to consider the special case of $Y := \mathcal{F}_A(X)$. In this case the desired property follows from the fact, which we prove in the next theorem, that ∂^A is equivalent to the dual of the Koszul differential, which we have already shown to have square zero in Theorem 3.2.16. Describing $C_*^A(\mathcal{F}_A(X))$ in terms of the Koszul complexes leads to the most convenient way of verifying that an operad is Koszul, see Corollary 3.5.8, and avoids the lengthy argument in the proof of Proposition 4.2.12 in [7]

Theorem 3.5.7. *For the free A-algebra generated by X the complex $C_*^A(\mathcal{F}_A(X))$ decomposes as a direct sum*

$$C_*^A(\mathcal{F}_A(X)) \cong \bigoplus_m C^A\{X,m\}.$$

Each of these subcomplexes is isomorphic to a dual Koszul complex $(\mathbf{K}(A^!)(m))^\#$ tensored with $X^{\otimes m}$:

$$\begin{aligned}
(C_*^A\{X,m\},\delta_{\{X,m\}}) &\cong \left(\mathbf{K}(A^!)(m)^\# \otimes (X^\#)^{\otimes m}, \delta_{\mathbf{K}}^\# \otimes id\right) && \text{for } m \geq 2\ ,\\
C^A\{X,1\} &= A(1)\otimes X \cong X && \text{for } m = 1\ .
\end{aligned}$$

Therefore,

$$\begin{aligned}
H_i(C^A\{X,m\}) &\cong H_i\left(\mathbf{K}(A^!)(m)^\#\right)\otimes (X^\#)^{\otimes m} && \text{for all } i \text{ if } m>1,\\
H_i\left(C^A\{X,1\}\right) &= 0 && \text{for } i>0 \text{ and}\\
H_0\left(C^A\{X,1\}\right) &\cong A(1)\otimes X \cong X.
\end{aligned}$$

Proof. For $n \geq 1$ the degree $n-1$ term decomposes as

$$\begin{aligned}
C_{n-1}^A(\mathcal{F}_A(X)) &= (A^!(n)^\# \otimes \mathrm{sgn}_n)\otimes_{K[n,n]} \mathcal{F}_A(X)^{\otimes n}\\
&\cong \bigoplus_{\{1\leq m_1,\dots,1\leq m_n\}} (A^!(n)^\# \otimes \mathrm{sgn}_n)\otimes_{K[n,n]} \Big((A(m_1)\otimes_{K[m_1,m_1]} X^{\otimes m_1})\otimes\cdots\\
&\qquad\qquad \cdots\otimes (A(m_n)\otimes_{K[m_n,m_n]} X^{\otimes m_n})\Big)\\
&\cong \bigoplus_{1\leq m_1,\dots,1\leq m_n} (A^!(n)^\# \otimes \mathrm{sgn}_n)\otimes_{K[n,n]} \times\\
&\qquad \Big((A(m_1)\otimes\cdots\otimes A(m_n))\otimes_{K[m_1,m_1]\otimes\cdots\otimes K[m_n,m_n]} X^{\otimes(m_1+\cdots+m_n)}\Big)\\
&\cong \bigoplus_{1\leq m} (A^!(n)^\# \otimes \mathrm{sgn}_n)\otimes_{K[n,n]} \left(A[n,m]\otimes_{K[m,m]} X^{\otimes m}\right)\\
&\cong \bigoplus_{1\leq m} C_{n-1}^A\{X,m\}
\end{aligned}$$

where

$$C_{n-1}^A\{X,m\} := (A^!(n)^\# \otimes \mathrm{sgn}_n)\otimes_{K[n,n]}\left(A[n,m]\otimes_{K[m,m]} X^{\otimes m}\right) \cong (\mathbf{K}(A^!)(m)^n)^\# \otimes_{K[m,m]} X^{\otimes m}.$$

The differential $C^A_n\{X,m\} \longrightarrow C^A_{n-1}\{X,m\}$ acts only on the factor $(A^!(n)^\# \otimes \mathrm{sgn}_n) \otimes_{K[n,n]} A[n,m]$ by

$$\begin{aligned}(A^!(n)^\# \otimes \mathrm{sgn}_n) \quad & \otimes_{K[n,n]} A[n,m] \\ & \longrightarrow \ (A^!(n-1)^\# \otimes_{K[n-1,n-1]} A^![n-1,n]^\# \otimes \mathrm{sgn}_n) \otimes_{K[n,n]} A[n,m] \\ & \longrightarrow \ (A^!(n-1)^\# \otimes_{K[n-1,n-1]} \mathrm{sgn}_{n-1} \otimes A^![n-1,n]^\#) \otimes_{K[n,n]} A[n,m] \\ & \longrightarrow \ (A^!(n-1)^\# \otimes \mathrm{sgn}_{n-1}) \otimes_{K[n-1,n-1]} (A[n-1,n] \otimes_{K[n,n]} A[n,m]) \\ & \longrightarrow \ (A^!(n-1)^\# \otimes \mathrm{sgn}_{n-1}) \otimes_{K[n-1,n-1]} A[n-1,m]\end{aligned}$$

which is the dual of the Koszul differential. This proves the first part of the theorem. The second part follows immediately. ■

Corollary 3.5.8. *A quadratic operad A is Koszul if and only if the A-algebra homology of the free A-algebra generated by X vanishes in all positive degrees and equals X in dimension 0.*

Proof. According to the previous theorem the vanishing condition stated in the corollary is equivalent to the exactness of all the Koszul complexes $\mathbf{K}(A^!)(m)$ for $m \geq 2$. By Theorem 3.44, this is equivalent to the fact that A is a Koszul operad. ■

Example 3.5.9. Let $K = \mathbf{k}$ and X an Ass-algebra, then since $Ass^!(n) = Ass(n) = \mathbf{k}[\Sigma_n]$, we have

$$C^{Ass}_{n-1}(X) = (\mathbf{k}[\Sigma_n] \otimes \mathrm{sgn}_n) \otimes_{\mathbf{k}[\Sigma_n]} X^{\otimes n} \cong \mathbf{k}[\Sigma_n] \otimes_{\mathbf{k}[\Sigma_n]} \otimes X^{\otimes n} \cong X^{\otimes n}.$$

The differential is the same as the Hochschild homology differential for an associative algebra. It is well known that for the free associative algebra on a vector space V

$$HH_t(\mathcal{F}_{Ass}(V)) = \begin{cases} 0 & \text{for } t \geq 1 \\ V & \text{for } t = 0 \end{cases}$$

Therefore, by Corollary 3.5.8, Ass is a Koszul operad.

Example 3.5.10. Let $K = \mathbf{k}$. Since $Lie^!(n) = Com(n) = \mathbb{1}$ the trivial representation of Σ_n

$$C^{Lie}_{n-1}(X) = (\mathbb{1} \otimes \mathrm{sgn}_n) \otimes_{\mathbf{k}[\Sigma_n]} X^{\otimes n} \cong \mathrm{sgn}_n \otimes_{\mathbf{k}[\Sigma_n]} X^{\otimes n} \cong \bigwedge{}^n X.$$

The differential is the Chevalley-Eilenberg differential. It is well known that for the free Lie algebra on a vector space V

$$H^{CE}_t(\mathcal{F}_{Lie}(V)) = \begin{cases} 0 & \text{for } t \geq 1 \\ V & \text{for } t = 0 \end{cases}$$

Again by Corollary 3.5.8 Lie is a Koszul operad.

Since an operad A is Koszul if and only if $A^!$ is Koszul, this shows that Com is Koszul. For the identification of the cohomology of an operad algebra with the standard cohomology theories, see Balavoine's thesis.

References.

[1] M. Barr, Cohomology in Tensored Categories, in M. Kelly and S. Eilenberg, eds., *Conference on Categorical Algebra*, Springer-Verlag, Berlin, Heidelberg, New York (1966) 344-354.

[2] M. Barr and J. Beck, Homology and Standard Constructions, in *Seminar on Triples and Categorical Homology Theory,* LNM 80, Springer-Verlag, New York (1969) 245-335.

[3] H.-J. Baues, M. Jibladze, and A. Tonks, Cohomology of Monoids in Monoidal Categories, in J.-L. Loday, et. al. eds, *Operads: Proceedings of Renaissance Conferences*, Contemporary Math. **202**, Am. Math. Soc. (1997) 137-166.

[4] A. Beilinson, V. Ginzburg, and V. Schechtman, Koszul Duality, *J. Geom. Phys.* bf 5 (1988) 317-350.

[5] S. Eilenberg, and J. C. Moore, Adjoint Functors and Triples, *Ill. J. Math.* **9** (1965) 381-389.

[6] M. Gerstenhaber, The Cohomology Structure of an Associative Ring, *Annals of Math.* **78** (1963) 268-288.

[7] V. Ginzburg, and M. Kapranov, Koszul Duality for Operads, *Duke Math. J.* **76** (1994) 203-272.

[8] E. Getzler, and J. D. S. Jones, Operads, Homotopy Algebra, and Iterated Integrals for Double Loop Spaces, Preprint (1994).

[9] P. Hilton, and U. Stammbach, *A Course in Homological Algebra* Springer Graduate Texts

[10] A. Joyal Une Théorie combinatoire des séries formelle Adv. Math **?**(?)

[11] G.M. Kelly, On Mac Lane's conditions for coherence of natural associativities, commutativities, etc. *J. Algebra* **1** (1964) 397-402.

[12] I. Kriz, J. P. May, Operads, Algebras, *Mod*ules, and Motives, *Astérisque***233** (1995).

[13] J.-L. Loday, La Renaissance des Opérades, *Séminaire Bourbaki***792** (1994) 47-74.

[14] S. Mac Lane, Natural Associativity and Commutativity, *Rice University Studies* **49** (1963) 28-46.

[15] —, Categorical Algebra, *Bull. Am. Math. Soc.* **71** (1965) 40-106.

[16] M. Markl, *Mod*els for Operads, *Communications in Algebra* **24** (1996) 1471-1500.

[17] J. P. May, The Geometry of Iterated Loop Spaces, LNM **271**, Springer-Verlag, New York (1972).

[18] S. B. Priddy, Koszul Resolutions, *Trans. Am. Math. Soc.***152** (1970) 39-60.

[19] S. Shnider and S. Sternberg, *Quantum Groups, From coalgebras to Drinfel'd Algebras* International Press, Cambridge, Mass. (1994)

[20] S. Shnider and D. Van Osdol, Operads as abstract algebras, and the Koszul property, *Jour. Pure. Appl. Alg.*, **143** (1999) 381-407.

[21] V. A. Smirnov, Cohomology of B- and co-B-construction, preprint in Russian.

[22] J. D. Stasheff, Homotopy Associativity of H - spaces I, II, *Trans. Am. Math. Soc.* **108**, (1968) 457-470.

3.6. Application: Minimal models and homotopy algebras.

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In this section we study homotopy properties of the category of differential graded operads which are intimately related to the nature of strongly homotopy algebras. The approach presented here, pioneered in [81], received recently a new impetus from the work of Kontsevich [65], Tamarkin and Tsygan [125], Voronov [132], and others related to the Formality Conjecture (see also Section **??**).

We introduce weak equivalence as the relation among differential graded operads generated by maps that induce isomorphism of cohomology – to be compared to weak homotopy equivalence in classical homotopy theory [114]. We show that there is a special class of differential graded operads, called minimal operads (Definition 3.6.5), with the property that they are isomorphic if and only if they are weakly equivalent (Theorem 3.6.7). We also show that, under some mild assumptions, each differential graded operad has a minimal model (Theorem 3.6.14), unique up to an isomorphism and functorial 'up to homotopy' (Theorem 3.6.15). This is the special cofibrant resolution mentioned in Section **??**. Finally, we define strongly homotopy algebras as algebras over these minimal operads. As an application of homotopy functoriality of the minimal model, we show that minimal models of Hopf operads admit a homotopy Hopf operad structure and interpret this Hopf structure in terms of corresponding algebras.

All algebraic objects in this section are defined over a fixed field $\mathbf{k}$ of characteristic zero. By an operad we mean in this section an operad in the category of differential graded $\mathbf{k}$-vector spaces.

The material of this section was inspired by rational homotopy theory. The statements and proofs are analogs of the corresponding classical results of rational homotopy theory as presented, for example, in [73] or [27]. The situation with operads is, however, in a sense easier than in rational homotopy theory, because here we have one more grading by the 'arity,' preserved by the differential. This may, in some cases, simplify inductive arguments. Most of the material of this section appeared in [81].

Definition 3.6.1. *A* quasi-isomorphism *(abbreviated as* quism *and sometimes called also an* elementary equivalence*) is a morphism $u : \mathcal{S} \to \mathcal{Q}$ of operads that induces an isomorphism of the homology, $H(u) : H(\mathcal{S}) \xrightarrow{\cong} H(\mathcal{Q})$.*

A weak equivalence *is the equivalence relation generated by the elementary equivalences. Weakly equivalent operads are said to have the same* weak homotopy type*.*

It is immediate from the definition that operads $\mathcal{S}$ and $\mathcal{Q}$ are weakly equivalent if and only if they are connected by a chain

$$\mathcal{S} \longleftarrow \mathcal{P}_1 \longrightarrow \mathcal{P}_2 \longleftarrow \cdots \longrightarrow \mathcal{P}_{s-1} \longleftarrow \mathcal{P}_s \longrightarrow \mathcal{Q} \tag{3.79}$$

of elementary equivalences. We are going to introduce minimal operads, but we need some auxiliary notions first.

Definition 3.6.2. *Let $\mathcal{P}$ be an operad with $\mathcal{P}(1) = \mathbf{k}$. Define the* decomposables $D\mathcal{P} = \{D\mathcal{P}(n)\}_{n\geq 1}$ *of the operad $\mathcal{P}$ to be the subcollection of $\mathcal{P}$ consisting of elements*

$$\gamma_{\mathcal{P}}(p, p_1, \ldots, p_k),\ p \in \mathcal{P}(k),\ p_i \in \mathcal{P}(n_i),\ 1 \leq i \leq k,$$

such that at least two of $k, n_1, \ldots, n_k$ are greater or equal to 2.

It is immediate to see that $D\mathcal{P}$ is an ideal in $\mathcal{P}$. A less formal way to define the decomposables, mimicking the definition of decomposables of an augmented algebra, is to introduce another ideal $\mathcal{P}^+$ by

$$\mathcal{P}^+(n) := \begin{cases} \mathcal{P}(n), & \text{for } n \geq 2, \text{ and} \\ 0, & \text{for } n = 0. \end{cases}$$

and then simply say that $D\mathcal{P} := (\mathcal{P}^+)^2$. One must though interpret this definition with care, since the γ-operations are not quadratic and $(\mathcal{P}^+)^2$ has, formally speaking, no meaning.

The above analogy can be made even more compelling if we introduce *augmented* operads as operads $\mathcal{P}$ equipped with a homomorphism $\epsilon : \mathcal{P} \to \mathbf{1}$ (the *augmentation*) to the trivial operad $\mathbf{1} = \{\mathbf{1}(n)\}_{n\geq 1}$ defined by

$$\mathbf{1}(n) := \begin{cases} \mathbf{k}, & \text{for } n = 1, \text{ and} \\ 0, & \text{for } n > 1. \end{cases}$$

Operads with $\mathcal{P}(1) = \mathbf{k}$ are clearly augmented, with an augmentation defined in an obvious manner. Decomposables of such an operad are then indeed the 'second power' of the augmentation ideal $\mathcal{P}^+ = \mathrm{Ker}(\epsilon)$.

To give an example, recall **??** that the free operad $\Gamma(E)$ decomposes as a direct sum indexed by trees, the decomposables $D\Gamma(E)$ then correspond to summands over trees with at least two (inner) vertices.

Let us make another useful definition related to decomposables.

Definition 3.6.3. *Let $\mathcal{P}$ be an operad with $\mathcal{P}(1) = \mathbf{k}$. The* indecomposables *of the operad $\mathcal{P}$ are defined to be the collection $Q\mathcal{P} = \{Q\mathcal{P}(n)\}_{n\geq 1}$ with $Q\mathcal{P}(n) := \mathcal{P}^+(n)/D\mathcal{P}(n)$.*

Example 3.6.4. Let us consider the free operad $\Gamma(E)$ on a collection E. Then E is a subcollection of $\Gamma(E)^+$ and the composition

$$E \hookrightarrow \Gamma(E) \xrightarrow{\text{proj}} Q\Gamma(E) \tag{3.80}$$

is an isomorphism. We will use (3.80) to identify $Q\Gamma(E)$ with the collection of generators E.

Here is the central definition of this section.

Definition 3.6.5. *A* minimal operad *is a differential graded operad $\mathfrak{M}$ of the form $\mathfrak{M} = (\Gamma(M), \partial)$, where $\Gamma(M)$ is the free operad* **??** *on a collection $M = \{M(n)\}_{n\geq 1}$ with $M(1) = 0$, and the differential ∂ is minimal in the sense that the image $\partial(M)$ of M consists of decomposable elements of the free operad $\mathcal{F}(M)$: $\partial(M) \subset D\Gamma(M)$.*

Example 3.6.6. Consider the dual operad $\mathbf{D}(\mathcal{S}) = (\mathbf{D}(\mathcal{S}), \partial_{\mathbf{D}})$ of a differential operad $\mathcal{S} = (\mathcal{S}, \partial_{\mathcal{S}})$ whose definition we recalled in **??**. The differential $\partial_{\mathbf{D}}$ consists of two pieces, $\partial_{\mathbf{D}} = \delta + \delta_{\mathcal{S}}$, where the 'inner' part $\delta_{\mathcal{S}}$ is induced by the dual of $\partial_{\mathcal{S}}$. We conclude that the dual operad $\mathbf{D}(\mathcal{S})$ is minimal if and only if $\partial_{\mathcal{S}} = 0$, i.e. if $\mathcal{S}$ has trivial differential.

The first important statement of this section says that minimal operads have the same weak homotopy type if and only if they are isomorphic.

Theorem 3.6.7. *Minimal operads are isomorphic if and only if they are weakly equivalent.*

The theorem will follow from a sequence of lemmas and propositions. Some proofs will contain inductive arguments, for which we need the following notation.

Given a collection $E = \{E(n)\}_{n\geq 1}$, for $k \geq 1$ we denote by $E(<\!k) = \{E(<\!k)(n)\}_{n\geq 1}$ the collection defined by

$$E(<\!k)(n) := \begin{cases} E(n), & \text{for } n < k, \text{ and} \\ 0, & \text{for } n \geq k. \end{cases}$$

Similarly, the collection $E(\leq\!k) = \{E(\leq\!k)(n)\}_{n\geq 1}$ is defined by

$$E(\leq\!k)(n) := \begin{cases} E(n), & \text{for } n \leq k, \text{ and} \\ 0, & \text{for } n > k. \end{cases}$$

Let $\mathfrak{M} = (\Gamma(M), \partial)$ be a minimal operad. Let us denote by

$$\mathfrak{M}^{(n)} := (\Gamma(M(\leq\!n)), \partial_n) \tag{3.81}$$

the differential suboperad of $\mathfrak{M}$ generated by $M(\leq\!n)$, with the inclusion $i : \mathfrak{M}^{(n)} \hookrightarrow \mathfrak{M}$. We are going to write an exact sequence relating the homologies $H(\mathfrak{M})$ and $H(\mathfrak{M}^{(n)})$. This sequence contains, besides the map $H(i)$ induced by the inclusion i, also maps j and δ defined as follows. The map

$$j : H(\mathfrak{M})(n\!+\!1) \to M(n\!+\!1) \tag{3.82}$$

is related to the decomposition $\mathfrak{M}(n+1) = \mathfrak{M}^{(n)}(n+1) \oplus M(n+1)$ which enables us to write each cycle $z \in Z(\mathfrak{M})(n+1)$ uniquely as $z = \omega + v$, with $\omega \in \mathfrak{M}^{(n)}(n+1)$ and $v \in M(n+1)$. We then put $j([x]) := v$. Since, by the minimality, all boundaries are decomposable, v depends on the homology class of x only and j is a well-defined.

The map $\delta : M(n+1) \to H(\mathfrak{M}^{(n)})(n+1)$ of degree -1 is defined by $\delta(v) := [\partial(v)]$, the homology class of $\partial(v) \in Z(\mathfrak{M}^{(n)})(n+1)$ in $H(\mathfrak{M}^{(n)})(n+1)$. The element $\partial(v)$ belongs to $\mathfrak{M}^{(n)}(n+1)$ because the differential ∂ is minimal.

Lemma 3.6.8. *The sequence of differential graded vector spaces*

$$H(\mathfrak{M}^{(n)})(n+1) \xrightarrow{H(i)} H(\mathfrak{M})(n+1) \xrightarrow{j} M(n+1) \xrightarrow{\delta} H(\mathfrak{M}^{(n)})(n+1) \xrightarrow{H(i)} H(\mathfrak{M})(n+1)$$

is exact, for any $n \geq 1$.

Proof. If $j([x]) = 0$, then necessarily $x \in Z(\mathfrak{M}^{(n)})(n+1)$, thus $[x] \in \mathrm{Im}(H(i))$, which is the exactness at the second term.

The condition $\delta(v) = 0$ means that there exists an element $\omega \in \mathfrak{M}^{(n)}(n+1)$ such that $\partial(\omega + v) = 0$. This implies $v = j[\omega + v]$ and the exactness at the middle term follows.

Let $z \in Z(\mathfrak{M}^{(n)})(n+1)$. Then $H(i)(z) = 0$ means that z is a boundary in $\mathfrak{M}$. Since $\mathfrak{M}(n+1) = \mathfrak{M}^{(n)}(n+1) \oplus M(n+1)$, $z = \partial\omega + \partial v$ for some $\omega \in \mathfrak{M}^{(n)}(n+1)$ and $v \in M(n+1)$, which means that $[z] = \delta(v)$. The exactness at the fourth term is also proved. ■

We are already able to prove the following important proposition.

Proposition 3.6.9. *A map $\phi : \mathfrak{M} \to \mathfrak{N}$ between minimal operads is an isomorphism if and only if it is a quasi-isomorphism.*

Proof. An isomorphism is plainly a quasi-isomorphism. Let us prove the opposite implication, that is, that a quasi-isomorphism $\phi : \mathfrak{M} \to \mathfrak{N}$ is an isomorphism. We prove, by induction, that the restriction $\phi_n : \mathfrak{M}^{(n)} \to \mathfrak{N}^{(n)}$, where $\mathfrak{M}^{(n)}$ and $\mathfrak{N}^{(n)}$ are defined as in (3.81) is an isomorphism for each n. This will imply, since $\phi = \varinjlim \phi_n$, that ϕ is an isomorphism, too.

Because $H(\mathfrak{M})(2) \cong M(2)$ and $H(\mathfrak{N})(2) \cong N(2)$, the map $\phi_2 : \mathfrak{M}^{(2)} \to \mathfrak{N}^{(2)}$ is plainly an isomorphism. Suppose we have already proved that ϕ_n is an isomorphism. Let us consider the following commutative diagram:

$$\begin{tikzcd}
H(\mathfrak{M}^{(n)})(n+1) \arrow[r] \arrow[d, "H(\phi_n)(n+1)"] & H(\mathfrak{M})(n+1) \arrow[r] \arrow[d, "H(\phi)(n+1)"] & M(n+1) \arrow[r] \arrow[d, "\overline{\phi}(n+1)"] & H(\mathfrak{M}^{(n)})(n+1) \arrow[r] \arrow[d, "H(\phi_n)(n+1)"] & H(\mathfrak{M})(n+1) \arrow[d, "H(\phi)(n+1)"] \\
H(\mathfrak{M}^{(n)})(n+1) \arrow[r] & H(\mathfrak{M})(n+1) \arrow[r] & M(n+1) \arrow[r] & H(\mathfrak{M}^{(n)})(n+1) \arrow[r] & H(\mathfrak{M})(n+1)
\end{tikzcd}$$

with the rows the exact sequences of Lemma 3.6.8 and $\overline{\phi}$ the induced map of generators. The map $H(\phi)(n+1)$ is an isomorphism because ϕ is a quasi-isomorphism, $H(\phi_n)(n+1)$ is an isomorphism by the induction assumption, thus we conclude from the five lemma [114] that $\overline{\phi}(n{+}1)$ is an isomorphism as well.

We proved that ϕ induces an isomorphism of the generators of $\mathfrak{M}^{(n+1)}$ and $\mathfrak{N}^{(n+1)}$, so ϕ_{n+1} is, by Lemma 3.6.21 of the Appendix, an isomorphism, too. The induction goes on. ∎

We are now in fact very close to a proof of Theorem 3.6.7 but, in order to put the last stone in place, we need to investigate lifting properties of minimal operads. To this end, we need an appropriate notion of homotopy.

For each operad $\mathcal{Q}$ and for each differential graded commutative $\mathbf{k}$-algebra R, the collection $R\otimes\mathcal{Q} = \{(R\otimes\mathcal{Q})(n)\}_{n\geq 1}$ defined by $(R\otimes\mathcal{Q})(n) := R\otimes_{\mathbf{k}}\mathcal{Q}(n)$, $n \geq 1$, is also an operad, with the composition maps defined by

$$\gamma_{R\otimes\mathcal{Q}}(a\otimes q; a_1\otimes q_1, \ldots, a_l\otimes q_l) := aa_1\cdots a_l\otimes\gamma_{\mathcal{Q}}(q, q_1, \ldots, q_l),$$

for $a, a_1, \ldots, a_l \in R$ and $q, q_1, \ldots, q_l \in \mathcal{Q}$. We consider, in particular, the differential graded commutative algebra $R := \mathbf{k}[t, \partial t] = \mathbf{k}[t]\otimes_{\mathbf{k}}E(\partial t)$, the tensor product of the polynomial algebra $\mathbf{k}[t]$ on the generator t of degree zero with the exterior algebra $E(\partial t)$ on the generator ∂t of degree -1. There are maps $\rho_i : \mathbf{k}[t, \partial t]\otimes\mathcal{Q} \to \mathcal{Q}$, $i = 0, 1$, defined as follows.

Each $x \in (\mathbf{k}[t, \partial t]\otimes\mathcal{Q})(n)$ is a sum of elements of the form $p(t)q_1 + r(t)\partial t\, q_2$, where $p(t), r(t) \in \mathbf{k}[t]$, $q_1, q_2 \in \mathcal{Q}(n)$. We then define

$$\rho_i(p(t)q_1 + r(t)\partial t\, q_2) := p(i)q_1,\ \ i = 0, 1$$

(the coefficient $p(i)$ is a scalar, $p(i) \in \mathbf{k}$). It is easy to see that the maps ρ_i are homomorphisms of operads. We define yet another map, $\textstyle\int : \mathbf{k}[t, \partial t]\otimes\mathcal{Q} \to \mathcal{Q}$, by

$$\textstyle\int(p(t)q_1 + r(t)\partial t\, q_2) := \left(\int_0^1 r(t)dt\right) q_1.$$

The map $\int$ is a degree $+1$ map of collections. The following definition is the first step towards a proper notion of homotopy.

Definition 3.6.10. *Two homomorphisms $f_0, f_1 : \mathcal{S} \to \mathcal{Q}$ of differential graded operads are* elementarily homotopic *if there exists a homomorphism $F : \mathcal{S} \to \mathbf{k}[t, \partial t]\otimes\mathcal{Q}$ such that $f_i = \rho_i \circ F$, $i = 1, 2$. The homomorphism F is called an* elementary homotopy *between f_0 and f_1.*

Let us show that elementary homotopy is a symmetric and reflexive relation. Each operadic homomorphism $f : \mathcal{S} \to \mathcal{Q}$ is clearly elementarily homotopic to itself, the elementary homotopy being the map $\tilde{f} : \mathcal{S} \to \mathbf{k}[t, \partial t] \otimes \mathcal{Q}$ given by

$$x \ni \mathcal{C}(n) \longmapsto 1 \otimes f(n)(x) \in \mathbf{k}[t, \partial t] \otimes \mathcal{Q}(n). \tag{3.83}$$

Let $\chi : \mathbf{k}[t, \partial t] \to \mathbf{k}[t, \partial t]$ be an endomorphism given by

$$\chi(t) := (1 - t) \text{ and } \chi(\partial t) := -\partial t.$$

If $F : \mathcal{S} \to \mathbf{k}[t, \partial t] \otimes \mathcal{Q}$ is an elementary homotopy between f_0 and f_1, then $(\chi \otimes \mathbb{1}_{\mathcal{Q}})(F)$ is an elementary homotopy between f_1 and f_0, as can be verified easily.

But it is *not* true that elementary homotopy is transitive. In fact, one can prove that it is transitive if the source object is cofibrant in an appropriate sense, but we will not need this result. In general, one must introduce homotopy as the equivalence relation generated by elementary homotopy.

Definition 3.6.11. *Operadic morphisms $f, g : \mathcal{S} \to \mathcal{Q}$ are* homotopic*, if there exists a sequence $\{\xi_i : \mathcal{S} \to \mathcal{Q}\}_{i=0}^{s}$ of operadic morphisms such that $f = \xi_0$, $\xi_s = g$ and ξ_i is elementarily homotopic to ξ_{i+1} for each $0 \leq i \leq s-1$. The* homotopy *between f and g is the corresponding chain of elementary homotopies.*

We claim that, if f_0, f_1 are homotopic in the sense of Definition 3.6.11, then the chain maps $f_0(n)$ and $f_1(n)$ are homotopic as chain maps, for each $n \geq 1$. It is clearly enough to verify this statement for the case when f_0 and f_1 are elementarily homotopic, with an elementary homotopy F.

Let us define a degree $+1$ map $G : \mathcal{S} \to \mathcal{Q}$ by $G := \int \circ F$. Let us show that that $G(n) : \mathcal{S}(n) \to \mathcal{Q}(n)$ is a chain homotopy between $f_0(n)$ and $f_1(n)$.

The nth component $F(n) : \mathcal{S}(n) \to \mathbf{k}[t, \partial t] \otimes \mathcal{Q}(n)$ of the homotopy F acts on $x \in \mathcal{S}(n)$ as

$$F(n)(x) = \sum p_i(t) A^i(x) + \sum r_j(t) \partial t \; B^j(x),$$

for some linear Σ_n-equivariant maps $A^i, B^j : \mathcal{S}(n) \to \mathcal{Q}(n)$ and polynomials $p_i, q_i \in \mathbf{k}[t]$. It is easy to see that $F(n)$ commutes with the differentials if and only if

$$\begin{aligned} \sum p_i(t)(A^i(\partial_{\mathcal{S}} x) - \partial_{\mathcal{Q}} A^i(x)) &= 0 \text{ and} \\ \sum r_j(t)(B^j(\partial_{\mathcal{S}}(x)) + \partial_{\mathcal{Q}}(B^j(x))) &= \sum \frac{dp_i(t)}{dt} A^i(x). \end{aligned} \tag{3.84}$$

Then $G(n)(x) = \left(\int_0^1 \sum r_j(t)dt\right) B^j(x)$ and

$$\begin{aligned}(\partial_{\mathcal{Q}} G(n) + G(n)\partial_{\mathcal{S}})(x) &= \int_0^1 \sum r_j(t)(B^j(\partial_{\mathcal{S}}(x)) + \partial_{\mathcal{Q}}(B^j(x)))dt = \text{ (by (3.84))} \\ &= \int_0^1 \sum \dot{p}_i(t) A^i(x)dt = \sum (p_i(1) - p_i(0))A^i(x) \\ &= f_1(n)(x) - f_0(n)(x).\end{aligned}$$

In particular, if f_0, f_1 are homotopic in the sense of Definition 3.6.11, then $H(f_0) = H(f_1)$. The converse is, of course, not true.

Let us formulate the following important lifting property of minimal operads. It suggests that minimal operads are cofibrations of a certain closed monoidal structure on the category of operads. This is indeed true [48], but we will not follow that direction in this book.

Theorem 3.6.12. *For each quism $\phi : \mathcal{S} \to \mathcal{Q}$ and for each morphism $f : \mathfrak{M} \to \mathcal{Q}$ of a minimal operad $\mathfrak{M}$ into $\mathcal{Q}$, there exists a map $h : \mathfrak{M} \to \mathcal{S}$ such that $\phi \circ h$ and f are homotopic:*

$$\begin{tikzcd} & \mathcal{S} \arrow[d, "\phi"] \\ \mathfrak{M} \arrow[ur, dotted, "h"] \arrow[r, "f"'] \arrow[ur, phantom, "\sim", shift right=3ex] & \mathcal{Q} \end{tikzcd}$$

where the symbol $\sim$ means that the diagram commutes up to homotopy only. The lift h is unique up to homotopy.

The proof is given in the Appendix to this section.

Definition 3.6.13. *Let $\mathcal{S}$ be a differential graded operad. A* minimal model *of $\mathcal{S}$ is a minimal operad $\mathfrak{M}$ together with a quasi-isomorphism $\alpha : \mathfrak{M} \to \mathcal{S}$.*

The assumption $H(S)(1) = \mathbf{k}$ in the following theorem plays the role of a simple connectivity assumption in rational homotopy theory. As in rational homotopy theory, it can be replaced by a kind of nilpotency, but we will not need this generality in the book.

Theorem 3.6.14. *Each differential graded operad $\mathcal{S} = (\mathcal{S}, \partial_{\mathcal{S}})$ such that $H(S)(1) = \mathbf{k}$ admits a minimal model $\alpha : \mathfrak{M} \to \mathcal{S}$.*

Proof. The proof is taken almost literally from [81] where we already translated a proof from [73] to operads. We will construct the minimal model $\mathfrak{M} = (\Gamma(M), \partial)$ and the quasi-isomorphism $\alpha : (\Gamma(M), \partial) \to (\mathcal{S}, \partial_{\mathcal{S}})$ inductively.

Let $M(2) := H(\mathcal{S})(2)$ and let $s(2) : M(2) \to Z(\mathcal{S})(2) \subset \mathcal{S}(2)$ be an Σ_2-equivariant section (see the Appendix) of the projection $\mathrm{cl} : Z(\mathcal{S})(2) \to H(\mathcal{S})(2)$ (here and later, cl will denote the projection of a cycle onto its homology class). Define a differential ∂_2 on $\Gamma(M(2))$ and a map $\alpha_2 : (\Gamma(M(2), \partial_2) \to (\mathcal{S}, \partial_{\mathcal{S}})$ by

$$\partial_2 := 0 \text{ and } \alpha_2|_{M(2)} := s(2).$$

Let us denote $\mathfrak{M}^{(2)} := (\Gamma(M(2)), \partial_2)$. It is clear that $H(\alpha_2)(n) : H(\mathfrak{M}^{(2)})(n) \to H(\mathcal{S})(n)$ is bijective for $n = 1, 2$.

Suppose we have already constructed a minimal operad $\mathfrak{M}^{(n-1)} = (\Gamma(M(<n)), \partial_{n-1})$ with a morphism $\alpha_{n-1} : \mathfrak{M}^{(n-1)} \to \mathcal{S}$ such that the map

$$H(\alpha_{n-1})(k) : H(\mathfrak{M}^{(n-1)})(k) \to H(\mathcal{S})(k)$$

is an isomorphism for any $k \leq n-1$. We show that there is a Σ_n-space $M(n)$, an extension ∂_n of the differential ∂_{n-1} and an extension $\alpha_n : \mathfrak{M}^{(n)} := (\Gamma(M(\leq n)), \partial_n) \to (\mathcal{S}, \partial_{\mathcal{S}})$ of the map α_{n-1} such that

$$H(\alpha_n)(k) : H(\mathfrak{M}^{(n)})(k) \to H(\mathcal{S})(k)$$

is an isomorphism for any $k \leq n$.

Let

$$A(n) := H(\mathcal{S})(n)/(\mathrm{Im}(H(\alpha_{n-1}))(n),$$
$$\overline{B}(n) := \mathrm{Ker}(H(\alpha_{n-1}))(n) \text{ and } B(n) := \uparrow \overline{B}(n).$$

Let $s(n) : A(n) \to Z(\mathcal{S})(n)$ be an equivariant section of the composition

$$Z(\mathcal{S})(n) \xrightarrow{\mathrm{cl}} H(\mathcal{S})(n) \xrightarrow{\mathrm{proj}} A(n).$$

Let also $r'(n) : H(\mathfrak{M}^{(n-1)})(n) \to Z(\mathfrak{M}^{(n-1)})(n)$ be an equivariant section of the projection

$$\mathrm{cl} : Z(\mathfrak{M}^{(n-1)})(n) \to H(\mathfrak{M}^{(n-1)})(n)$$

and let $r(n) : \overline{B}(n) \to Z(\mathfrak{M}^{(n-1)})(n)$ be the composition

$$B(n) \xrightarrow{\downarrow} \overline{B}(n) \hookrightarrow H(\mathfrak{M}^{(n-1)})(n) \xrightarrow{r'(n)} Z(\mathfrak{M}^{(n-1)}(n)).$$

Since, by definition, $\alpha_{n-1}(r(n)(b))$ is, for $b \in B(n)$, homologous to zero in $H(\mathcal{S})$, there exists a linear Σ_n-equivariant degree zero map $\beta(n) : B(n) \to \mathcal{S}(n)$ such that

$$\alpha_{n-1}(r(n)(b)) = \partial_{\mathcal{S}}(\beta(n)(b)), \ b \in B(n). \tag{3.85}$$

Let us define $M(n) := A(n) \oplus B(n)$ and extend the differential ∂_{n-1} and the map α_{n-1} by

$$\partial_n|_{A(n)} := 0,\ \partial_n|_{B(n)} := r(n),\ \alpha_n|_{A(n)} := s(n) \text{ and } \alpha_n|_{B(n)} := \beta(n).$$

Then ∂_n is minimal, since

$$Z(\Gamma(M(<n)), \partial_{n-1})(n) \subset \Gamma(M(<n))(n) \subset D\Gamma(M(<n))(n).$$

It is also immediate to see that $\partial_n^2 = 0$. Let us show that $\alpha_n : \mathfrak{M}^{(n)} \to \mathcal{S}$ is a chain map, that is,

$$\alpha_n(\partial_n(a)) = \partial_{\mathcal{S}}(\alpha_n(a)), \text{ for } a \in A(n) \tag{3.86}$$

and

$$\alpha_n(\partial_n(b)) = \partial_{\mathcal{S}}(\alpha_n(b)), \text{ for } b \in B(n). \tag{3.87}$$

Since $\partial_n(a) = 0$, equation (3.86) reduces to $\partial_{\mathcal{S}}(\alpha_n(a)) = \partial_{\mathcal{S}}(s(n)(a)) = 0$ which is true, because $\mathrm{Im}(s(n)) \subset Z(S)(n)$. Equation (3.87) is a consequence of

$$\alpha_n(\partial_n(b)) = \alpha_n(r(n)(b)) = \alpha_{n-1}(r(n)(b)) = \partial_{\mathcal{S}}(\beta(n)(b)) = \partial_{\mathcal{S}}(\alpha_n(b))$$

which follows from (3.85).

The map $H(\alpha_n)(n) : H(\mathfrak{M}^{(n)})(n) \to H(\mathcal{S})(n)$ is, by construction, an epimorphism. Let us prove that it is a monomorphism.

Let $z \in Z(\mathfrak{M}^{(n)})(n)$ and write it in the form $z = a + b + \omega$, with $a \in A(n)$, $b \in B(n)$ and $\omega \in D\Gamma(M(<n))$. We show $\partial_n(z) = 0$ implies $b = 0$. By definition $\partial_n(a) = 0$, therefore

$$-\partial_n(b) = -r(n)(b) = \partial_n(\omega).$$

Since ω is decomposable, we know in fact that $\partial_n(\omega) = \partial_{n-1}(\omega)$, therefore $\partial_n(\omega)$ represents a trivial homological class in $H(\mathfrak{M}^{(n-1)})(n)$, thus $b = 0$.

If moreover $[z] \in \mathrm{Ker}(H(\alpha_n))(n)$, then $a = 0$, so $z \in Z(\mathfrak{M}^{(n-1)})(n)$ and so, in fact, $[z] \in \mathrm{Ker}(H(\alpha_{n-1}))(n)$. But then $[z] \in B(n)$, so $z = \partial_n(\uparrow [z])$ by the construction of ∂_n which implies that z represents a trivial homology class in $H(\mathfrak{M}^{(n)})$.

We verified that $\alpha_n : \mathfrak{M}^{(n)} \to \mathcal{S}$ has the desired properties, and the induction may go on, giving rise to the minimal model $\mathfrak{M} := \varinjlim \mathfrak{M}^{(n)}$ and a quism $\alpha := \varinjlim \alpha_n$. ■

The following theorem is an easy corollary to Theorem 3.6.12.

Theorem 3.6.15. *Let $\mathcal{S}$ and $\mathcal{Q}$ be differential graded operads with $\alpha_{\mathcal{S}} : \mathfrak{M}_{\mathcal{S}} \to \mathcal{S}$ and $\alpha_{\mathcal{Q}} : \mathfrak{M}_{\mathcal{Q}} \to \mathcal{Q}$ their minimal models.*

Given a morphism $f : \mathcal{S} \to \mathcal{Q}$, there exists a map $\tilde{f} : \mathfrak{M}_{\mathcal{S}} \to \mathfrak{M}_{\mathcal{Q}}$ such that $f \circ \alpha_{\mathcal{S}}$ is homotopic to $\alpha_{\mathcal{Q}} \circ \tilde{f}$:

$$\begin{tikzcd} \mathfrak{M}_{\mathcal{S}} \arrow[r, "\tilde{f}"] \arrow[d, "\alpha_{\mathcal{S}}"'] \arrow[dr, phantom, "\sim"] & \mathfrak{M}_{\mathcal{Q}} \arrow[d, "\alpha_{\mathcal{Q}}"] \\ \mathcal{S} \arrow[r, "f"] & \mathcal{Q} \end{tikzcd}$$

The minimal model is unique up to isomorphism.

Proof. The first part of the theorem immediately follows from Theorem 3.6.12. To prove the second part, assume we have two minimal models of the same operad $\mathcal{S}$ and apply the first part of the theorem to construct a lift $\tilde{1\!\!1}_{\mathcal{S}}$ of the identity map $1\!\!1_{\mathcal{S}} : \mathcal{S} \to \mathcal{S}$. Since clearly $\tilde{1\!\!1}_{\mathcal{S}}$ is a homology isomorphism, it is, by Theorem 3.6.9, an isomorphism of the two minimal models. ■

Theorem 3.6.16. *Operads $\mathcal{S}$ and $\mathcal{Q}$ have the same weak homotopy type if and only if they have isomorphic minimal models.*

Proof. It is clear that two operads whose minimal models are isomorphic have the same weak homotopy type.

Let us prove the converse, that is, suppose we have a chain (3.79) of elementary equivalences that connect $\mathcal{P}$ and $\mathcal{Q}$ and prove that then their minimal models $\mathfrak{M}_{\mathcal{S}}$ and $\mathfrak{M}_{\mathcal{Q}}$ are isomorphic. To this end, consider the diagram

$$\begin{tikzcd} \mathfrak{M}_{\mathcal{S}} \arrow[d] \arrow[dr, phantom, "\sim"] & \mathfrak{M}_1 \arrow[l] \arrow[r] \arrow[d] \arrow[dr, phantom, "\sim"] & \mathfrak{M}_2 \arrow[d] & \cdots \arrow[l] \arrow[r] & \mathfrak{M}_{s-1} \arrow[d] \arrow[dr, phantom, "\sim"] & \mathfrak{M}_s \arrow[l] \arrow[r] \arrow[d] \arrow[dr, phantom, "\sim"] & \mathfrak{M}_{\mathcal{Q}} \arrow[d] \\ \mathcal{S} & \mathcal{P}_1 \arrow[l] \arrow[r] & \mathcal{P}_2 & \cdots \arrow[l] \arrow[r] & \mathcal{P}_{s-1} & \mathcal{P}_s \arrow[l] \arrow[r] & \mathcal{Q} \end{tikzcd}$$

in which the bottom row is the chain (3.79), vertical maps are the minimal models and the top row consists of homotopy lifts of the maps in the bottom row which exist, by Theorem 3.6.15. Since all maps in the top row are quisms, they are in fact isomorphisms and the theorem is proved. ■

Theorem 3.6.7 easily follows from Theorem 3.6.15 because a minimal operad is its own minimal model. The above results can be summarized by saying that there is exactly one (up to an isomorphism) minimal operad inside each weak homotopy type of operads. Let us introduce our concept of strongly homotopy algebras, see also [86] for a gentle pedagogical introduction to these objects.

Definition 3.6.17. *Let $\mathcal{S}$ be a differential graded operad. A* strongly homotopy $\mathcal{S}$-algebra*, or an* sh $\mathcal{S}$-algebra *for short, is an algebra over the minimal model $\mathfrak{M}_{\mathcal{S}}$ of $\mathcal{S}$.*

One often denotes strongly homotopy $\mathcal{S}$-algebras as $\mathcal{S}_\infty$-algebras, but there is one important exception, namely the operad $\mathcal{B}_\infty$ for 'Baues' algebras, introduced in [36], which is not minimal, that is, algebras over this operad are not strongly homotopy algebras in the sense of Definition 3.6.17 (and, moreover, there is no corresponding $\mathcal{B}$). See also Section **??** of this book.

Example 3.6.18. Each graded non-differential operad $\mathcal{P}$ can be considered as a differential operad with trivial differential. If such an operad is quadratic Koszul in the sense of **??** then, by **??**, the canonical map $\mathbf{D}(\mathcal{P}^!) \to \mathcal{P}$ from the dual operad of the quadratic dual $\mathcal{P}^!$ of $\mathcal{P}$ to $\mathcal{P}$ is a quasi-isomorphism. As we observed in Example 3.6.6, the operad $\mathbf{D}(\mathcal{P}^!)$ is minimal, thus the canonical map $\mathbf{D}(\mathcal{P}^!) \to \mathcal{P}$ is the minimal model of $\mathcal{P}$. By the uniqueness of Theorem 3.6.15, each minimal model of $\mathcal{P}$ is isomorphic to this one.

Recall that Ginzburg and Kapranov defined strongly homotopy $\mathcal{P}$-algebras for nondifferential quadratic Koszul operads precisely as algebras over $\mathbf{D}(\mathcal{P}^!)$. We see that in this particular case their definition coincides with Definition 3.6.17. Most existing strongly homotopy algebras are of this type, in particular, all 'classical' examples such as A_∞, L_∞, C_∞ and G_∞ algebras discussed in **??**. In fact, all 'natural' strongly homotopy algebras must be of this type, because all 'reasonable' strict algebraic structures are algebras over quadratic Koszul operads. We think that this reflects a deeper gnostic pinciple that strict algebras over operads which are not Koszul would manifest weird phenomena and thus they would be excluded from the list.

An 'unnatural' example which was not of Ginzburg-Kapranov type was, however, constructed in [81]. In the realm of *colored operads*, 'strongly homotopy algebras' typically represent *diagrams* of strongly homotopy algebras and their strongly homotopy (in an appropriate sense) morphisms, and examples which are not of Ginzburg-Kapranov type are abundant, see [86, 87].

Let us give an application of the technique developed in this section. Given a differential graded operad $\mathcal{S}$, we introduce the tensor product $\mathcal{S}\otimes\mathcal{S} = \{(\mathcal{S}\otimes\mathcal{S})(n)\}_{n\geq 1}$ to be the collection with $(\mathcal{S}\otimes\mathcal{S})(n) := \mathcal{S}(n)\otimes_{\mathbf{k}}\mathcal{S}(n)$ and the obvious operadic structure. The following definition was taken from [36].

Definition 3.6.19. *The operad $\mathcal{P}$ is a* Hopf operad *if there exists an operadic map $\Delta : \mathcal{S} \to \mathcal{S}\otimes\mathcal{S}$ (the* diagonal*) which is associative:*

$$(\Delta\otimes 1\!\!1)\Delta = (1\!\!1\otimes\Delta)\Delta : \mathcal{S}\longrightarrow\mathcal{S}\otimes\mathcal{S}\otimes\mathcal{S}.$$

Algebras over a Hopf operad form a strict monoidal category, with the monoidal structure $\square$ induced by the diagonal Δ as follows.

Let $A = (A, a_A)$ and $B = (B, a_B)$ be two $\mathcal{S}$-algebras, with structure maps $a_A : \mathcal{S} \to End_A$ and $a_B : \mathcal{S} \to End_B$. Then we define $A \square B$ to be the vector space $A \otimes B$ with the structure map $a_{A\otimes B} : \mathcal{S} \to End_{A\otimes B}$ given as the composition

$$\mathcal{S} \xrightarrow{\Delta} \mathcal{S}\otimes\mathcal{S} \xrightarrow{a_A\otimes a_B} End_A \otimes End_B \xrightarrow{j} End_{A\otimes B},$$

where $j : End_A \otimes End_B \to End_{A\otimes B}$ is the homomorphism with components

$$j(n) : (End_A \otimes End_B)(n) \to End_{A\otimes B}(n)$$

defined as the composition

$$\begin{aligned}(End_A \otimes End_B)(n) &= End_A(n) \otimes End_B(n) = Hom(A^{\otimes n}, A) \otimes Hom(B^{\otimes n}, B)\\ &\xrightarrow{\cong} Hom(A^{\otimes n} \otimes B^{\otimes n}, A\otimes B) \xrightarrow{\cong} Hom((A\otimes B)^{\otimes n}, A\otimes B) = End_{A\otimes B}.\end{aligned}$$

As an immediate application of Theorem 3.6.12 we get the following proposition.

Proposition 3.6.20. *The minimal model $\alpha_{\mathcal{S}} : \mathfrak{M} \to \mathcal{S}$ of a Hopf operad $\mathcal{S}$ admits a homotopy Hopf operad structure, that is, a diagonal $\tilde{\Delta} : \mathfrak{M} \to \mathfrak{M}\otimes\mathfrak{M}$ such that the maps $(\tilde{\Delta}\otimes \mathbb{1})\tilde{\Delta}$ and $(\mathbb{1}\otimes\tilde{\Delta})\tilde{\Delta}$ are homotopic as morphisms of operads in the sense of Definition 3.6.11.*

Proof. As an immediate consequence of the Künneth theorem, we see that the map $\alpha_{\mathcal{S}} \otimes \alpha_{\mathcal{S}} : \mathfrak{M}\otimes\mathfrak{M} \to \mathcal{S}\otimes\mathcal{S}$ is a quasi-isomorphism. The diagonal $\tilde{\Delta} : \mathfrak{M} \to \mathfrak{M}\otimes\mathfrak{M}$ is then the homotopy lift of $\Delta : \mathcal{S}\otimes\mathcal{S} \to \mathcal{S}$ which exists, by Theorem 3.6.12.

The homotopy coassociativity of $\tilde{\Delta}$ follows immediately from the uniqueness up to homotopy of the homotopy lift, since both $(\tilde{\Delta}\otimes\mathbb{1})\tilde{\Delta}$ and $(\mathbb{1}\otimes\tilde{\Delta})\tilde{\Delta}$ are lifts of the same map $(\Delta\otimes\mathbb{1})\Delta = (\mathbb{1}\otimes\Delta)\Delta : \mathcal{S} \to \mathcal{S}\otimes\mathcal{S}\otimes\mathcal{S}$. ■

The prominent example of a Hopf operad is the operad Ass for associative algebras, with the diagonal δ given by

$$Ass(n) \ni x \longmapsto x \otimes x \in Ass(n) \otimes Ass(n).$$

Proposition 3.6.20 then predicts the existence of a homotopy Hopf structure on the minimal model of Ass, that is, on the operad $\mathcal{A}_\infty$ for A_∞-algebras. An explicit example of such a diagonal was constructed by Saneblidze and Umble in [106].

3.6.1. Appendix.

The following lemma is necessary for the proof of Proposition 3.6.9.

Lemma 3.6.21. *Let $\phi : \Gamma(M) \to \Gamma(N)$ be a homomorphism of free operads and let $\overline{\phi} : M \to N$ be the induced map of the indecomposables (under the identification (3.80)). Suppose that $M(1) = N(1) = 0$. Then ϕ is an isomorphism of operads if and only if $\overline{\phi}$ is an isomorphism of collections.*

Proof. We prove the lemma by induction. Suppose that we have already proved that the restriction

$$\phi_{<n} : \Gamma(M(<n)) \to \Gamma(N(<n))$$

is an isomorphism for some $n \geq 3$ (for $n = 2$ the statement is trivial) we prove that then the restriction

$$\phi_n : \Gamma(M(\leq n)) \to \Gamma(N(\leq n))$$

is an isomorphism, too. To this end, it is clearly enough to verify that

$$\text{the restriction } \phi|_{M(n)} : M(n) \to \Gamma(N)(n) \text{ is a monomorphism} \tag{3.88}$$

and that

$$N(n) \subset \mathrm{Im}(\phi)(n). \tag{3.89}$$

It follows from the assumption $M(1) = N(1) = 0$ that the restriction $\phi|_{M(n)}$ decomposes as

$$\phi|_{M(n)}(v) = \overline{\phi}(v) + \phi^+(v), \text{ for } v \in M(n),$$

where $\phi^+(v) \in \Gamma(N(<n))(n)$. This immediately implies (3.88). To prove (3.89) we invoke the induction assumption to see that, for any $v \in M(n)$, there exists some $\omega \in \Gamma(M(<n))$ such that $\phi^+(v) = \phi(\omega)$. Then each $u \in N(n)$ is the image of $v - \omega$ with $v := \overline{\phi}^{-1}(u)$. ■

Equivariant section. Given two collections F, G and an epimorphism $\pi : F \to G$ (i.e. a morphism of collections such that $\pi(n) : F(n) \to G(n)$ is an epimorphism for each $n \geq 1$), there always exists a map of collections $s : G \to F$ such that $\pi \circ s = \mathbb{1}_G$. Indeed, we must show that there exists, for each $n \geq 1$, a Σ_n-equivariant map $s(n) : G(n) \to F(n)$ such that $\pi(n) \circ s(n) = \mathbb{1}_{G(n)}$. To construct this map, choose a arbitrary section $\tilde{s}(n) : G(n) \to F(n)$ of $\pi(n)$ and put, for $g \in G(n)$,

$$s(n)(g) := \sum_{\sigma \in \Sigma_n} \frac{1}{n!} [\tilde{s}(n)(g\sigma^{-1})]\sigma.$$

Observe that here the assumption $\mathrm{char}(\mathbf{k}) = 0$ is crucial. We will call such s an *equivariant section* of π.

Proof of Theorem 3.6.12. We use the fact that minimal operads are limits of special types of inclusions which we now introduce. Suppose we are given a differential graded operad $\mathcal{B} = (\mathcal{B}, \partial_{\mathcal{B}})$, a collection M and a degree -1 map $\tau : M \to \mathcal{B}$ of collections such that $\mathrm{Im}(\tau) \subset Z(\mathcal{B})$. Let us extend the differential $\partial_{\mathcal{B}}$ to a differential ∂_τ on the free product $\mathcal{B} * \Gamma(M)$ by $\partial_\tau(v) := \tau(v)$, for $v \in M$. Let us finally denote

$$\mathcal{B} *_\tau \Gamma(M) := (\mathcal{B} * \Gamma(M), \partial_\tau).$$

Definition 3.6.22. *A* principal extension *is an inclusion of differential graded operads of the form* $\iota : \mathcal{B} \hookrightarrow \mathcal{B} *_\tau \Gamma(M)$.

Observe that $\mathfrak{M}^{(n+1)} = \mathfrak{M}^{(n)} *_\tau \Gamma(M(n+1))$ (the notation of (3.81)) so each minimal operad is a direct limit of principal extensions. Theorem 3.6.12 will thus follow from the following lemma.

Lemma 3.6.23. *For each quasi-isomorphism* $\phi : \mathcal{S} \to \mathcal{Q}$*, for each principal extension* $\iota : \mathcal{B} \hookrightarrow \mathcal{B} *_\tau \Gamma(M)$ *and for each couple of maps* $f : \mathcal{B} \to \mathcal{S}$ *and* $g : \mathcal{B} *_\tau \Gamma(M) \to \mathcal{Q}$ *such that* $\phi \circ f$ *is homotopic to* $g \circ \iota$*, there exists an extension* f *to* $h : \mathcal{B} *_\tau \Gamma(M) \to \mathcal{S}$ *such that* $\phi \circ h$ *is homotopic to* g*:*

$$\begin{tikzcd}
\mathcal{B} \arrow[r, "f"] \arrow[d, "\iota"'] & \mathcal{S} \arrow[d, "\phi"] \\
\mathcal{B} *_\tau \Gamma(M) \arrow[r, "g"'] \arrow[ur, dotted, "h"] & \mathcal{Q} \arrow[ul, phantom, "\sim" near start]
\end{tikzcd}$$

Proof. Let us show first that we may in fact assume that the diagram in Lemma 3.6.23 commutes strictly, that is

$$g \circ \iota = \phi \circ f. \tag{3.90}$$

To be more precise, we show that there exists some $g' : \mathcal{B} *_\tau \Gamma(M) \to \mathcal{Q}$, homotopic to g, for which $g' \circ \iota = \phi \circ f$. This will follow from the following claim.

Claim. *Suppose we are given a principal extension* $\iota : \mathcal{B} \hookrightarrow \mathcal{B} *_\tau \Gamma(M)$ *and maps* $\alpha : \mathcal{B} \to \mathcal{Q}$ *and* $\beta : \mathcal{B} *_\tau \Gamma(M) \to \mathcal{Q}$ *such that* α *is elementarily homotopic to* $\beta \circ \iota$*. Then there exists a map* β'*, elementarily homotopic to* β*, for which* $\alpha = \beta' \circ \iota$*.*

To prove the claim, let $F : \mathcal{B} \to \mathbf{k}[t, \partial t] \otimes \mathcal{Q}$ be an elementary homotopy between $\beta \circ \iota$ and α, that is, $\rho_0 \circ F = \beta \circ \iota$, $\rho_1 \circ F = \alpha$. We extend F to an elementary homotopy $H : \mathcal{B} *_\tau \Gamma(M) \to \mathbf{k}[t, \partial t] \otimes \mathcal{Q}$ such that $\rho_0 \circ H = \beta$. The map $\beta' := \rho_1 \circ H$ then automatically satisfies $\beta' \circ \iota = \alpha$, because $\beta' \circ \iota = \rho_1 \circ H \circ \iota = \rho_1 \circ F = \alpha$. The situation is depicted in the following diagram:

$$
\begin{tikzcd}
\mathcal{Q} & & \mathcal{B} *_\tau \Gamma(M) \arrow[ll, "\beta"'] \arrow[dll, dotted, "H"'] \\
\mathbf{k}[t,\partial t]\otimes\mathcal{Q} \arrow[u, "\rho"] & & \mathcal{B} \arrow[ll, "F"'] \arrow[u, "\iota"']
\end{tikzcd}
$$

Let $K := \mathrm{Ker}(\rho_0)$. Since ρ_0 is a surjective quism, the collection K is acyclic. Choose an equivariant section $\sigma : Z(K) \to K$ of the epimorphism $\partial : K \to Z(K)$. It is immediate to see, for each $v \in M$, that $F\partial_\tau(v) - 1\otimes(\beta \circ \iota \circ \tau(v))$ is a cycle in K. So we can define the extension H by

$$H(1 * v) := 1\otimes(\beta(1 * v)) + \sigma(F\partial_\tau(v) - 1\otimes(\beta \circ \iota \circ \tau(v))).$$

Now $\rho_0 \circ H = \beta$ is verified easily, because the 'correction' $\sigma(F\partial_\tau(v) - 1\otimes(\beta\circ\iota\circ\tau(v)))$ belongs to the kernel K of ρ_0, by the definition of σ. It is equally easy to verify that H commutes with the differentials. The Claim is proved.

Let us turn our attention back to the diagram of Lemma 3.6.23. The homotopy commutativity of the lower right triangle means that we are given a sequence $\xi_1, \ldots, \xi_{s-1}$ of maps such that $g \circ \iota$ is elementarily homotopic to ξ_1, ξ_i is elementarily homotopic to ξ_{i+1} for $1 \le i < s-1$ and ξ_{s-1} is elementarily homotopic to $\phi \circ f$. Use the Claim (with $\alpha = \xi_1$ and $\beta = g$) to replace g by a homotopic map g_1 such that $g_1 \circ \iota = \xi_1$. Now $g_1 \circ \iota$ is elementarily homotopic to ξ_2. We may use the Claim again to replace g_1 by some g_2, homotopic to g_1, such that $g_2 \circ \iota = \xi_2$. Repeating this process $s-1$ times we end up with a $g' = g_{s-1}$, homotopic to the original g, such that $g' \circ \iota = \phi \circ f$. So we may indeed assume (3.90).

Before we begin to construct the extension h for Lemma 3.6.23, we need to verify that the image $\mathrm{Im}(f \circ \tau)$ consists of boundaries. Indeed, $(f \circ \tau)(v) \in Z(\mathcal{S})$ for $v \in M$ and the equation

$$(\phi \circ f \circ \tau)(v) = (g \circ \iota \circ \tau)(v) = g(\partial_\tau(v)) = \partial g(1 * v)$$

shows that $(\phi \circ f \circ \tau)(v)$ is a boundary. Since ϕ is a quism, $(f \circ \tau)(v)$ must be a boundary as well.

Let γ be an equivariant section of $\partial : \mathcal{S} \to \partial(\mathcal{S})$. As the first approximation to the extension h, consider an extension $\tilde{h} : \mathcal{B} *_\tau \Gamma(M) \to \mathcal{S}$ of f defined by

$$\tilde{h}(1 * v) := (\gamma \circ f \circ \tau)(v) \text{ for } v \in M.$$

It is easy to verify that $\tilde{h}$ commutes with the differentials. Now we must, by adding a correction term from $Z(\mathcal{S})$, modify $\tilde{h}$ to some h such that $\phi \circ h$ and g are homotopic.

Let $\tilde{g} : \mathcal{B} \to \mathbf{k}[t,\partial t]\otimes\mathcal{Q}$ be the trivial homotopy of $g \circ \iota$ to itself (compare (3.83)) defined by

$$\tilde{g}(b) := 1\otimes(g \circ \iota)(b) = 1\otimes(\phi \circ f)(b) \text{ for } b \in \mathcal{B}.$$

Let us extend $\tilde{g}$ naively to a 'homotopy' $\tilde{H} : \mathcal{B} *_\tau \Gamma(M) \to \mathbf{k}[t, \partial t] \otimes \mathcal{Q}$ between g and $\phi \circ \tilde{h}$ by

$$\tilde{H}(1 * v) := (1-t)g(1*v) + t(\phi \circ \tilde{h})(1 * v),\ \ v \in M.$$

Unfortunately, it is not true in general that this naive $\tilde{H}$ commutes with the differentials. Let us denote by δ the deviation from this being true,

$$\delta(v) = \partial \tilde{H}(1*v) - \tilde{g}(\tau(v)) = \partial t\, (\phi \circ \tilde{h} - g)(1*v).$$

Now we invoke (3.90) to verify that $((\phi \circ \tilde{h} - g)(1*v))$ is a cycle in $\mathcal{Q}$:

$$\partial_{\mathcal{Q}}(\phi \circ \tilde{h} - g)(1*v) = (\phi \circ \tilde{h} - g)(\tau(v)) = (\phi \circ f - g \circ \iota)(\tau(v)) = 0.$$

Since ϕ is a quism, there exist maps $\psi : M \to Z(\mathcal{S})$ and $\eta : M \to \mathcal{Q}$ of collections such that, for each $v \in M$,

$$(\phi \circ \psi)(v) - (\phi \circ \tilde{h} - g)(1*v) = (\partial_{\mathcal{Q}} \circ \eta)(v),\ \ v \in M. \tag{3.91}$$

Let h be the extension of f defined by

$$h(1*v) := \tilde{h}(1*v) - \psi(v).$$

Define then the homotopy $H : \mathcal{B} *_\tau \Gamma(M) \to \mathbf{k}[t, \partial t] \otimes \mathcal{Q}$ as the extension of $\tilde{g}$ satisfying

$$H(1*v) := (1-t)g(1*v) + t(\phi \circ \tilde{h})(1 \otimes v) + \partial t\, \eta(v).$$

It follows from (3.91) that the map H commutes with the differentials. It is clear that $\rho_0 \circ H = g$ and $\rho_1 \circ H = \phi \circ h$, therefore $\phi \circ h$ is homotopic to g as claimed.

We will not prove the uniqueness part of Theorem 3.6.12 here. The proof can be obtained by translating the arguments of [27, Theorem 3.7] into the operadic language. ■

Chapter 4

Geometry

4.1. Configuration Spaces Operads and Modules.

Version of April 26, 2000,
the second half needs a deep revision

The aim of this section is to provide the reader with a road map of the complicated world of operads and modules over these operads that are constructed from configurations of geometric objects (disks, cubes, points) in manifolds of various types.

Little disks operads. We have already seen examples of 'geometric' operads in this book. Let us recall **??** at least the little k-cubes operad $\mathcal{C}_k$ whose detailed definition we gave in **??**. The *little k-disks operad* $\mathcal{D}_k = \{\mathcal{D}_k(n)\}_{n\geq 1}$ is defined quite analogously, except that we use little disks instead of little cubes. This means that $\mathcal{D}_k(n)$ is, for $n \geq 1$, the space of all maps

$$d : \coprod_{1\leq s\leq n} D^k_s \to D^k$$

from the disjoint union of n numbered standard disks $D_1, \ldots, D_n$ to D^k, where

$$D^k := \{(x_1, \ldots, x_k) \in \mathbb{R}^k;\ x_1^2 + \cdots + x_k^2 \leq 1\},$$

such that d, when restricted to each disk, is a composition of translation and multiplication by a positive real number, and the images of the interiors of the disks are disjoint. The symmetric group acts by renumbering the disks. We may interpret $d \in \mathcal{D}_k(n)$ as D^k with n numbered, disjoint circular holes, since all information about d is kept by the shape of its image.

The operadic structure on the collection $\mathcal{D}_k = \{\mathcal{D}_k(n)\}_{n\geq 1}$ is given exactly as for little cubes **??**, so we will not give the definition here. We may informally visualize the composition $\gamma_{\mathcal{D}_k}(d; d_1, \ldots, d_n)$ as the gluing of the ith disk d_i into the ith hole of d, for $1 \leq i \leq n$.

Let us introduce a 'framed' version $f\mathcal{D}_k = \{f\mathcal{D}_k(n)\}_{n\geq 1}$ of the little disk operad $\mathcal{D}_k$. As in the unframed case, elements of $f\mathcal{D}_k(n)$ are maps from the disjoint union of n standard disks D^k to D^k, but here we admit *all* conformal linear maps, that is, maps that are compositions of translation, multiplication by a positive real number, and a rotation by an element of the orthogonal group $O(n)$. Now an element $f \in f\mathcal{D}_k(n)$ cannot be described just as the disk D^k with n numbered holes, but each hole must also be decorated by a *frame* that encodes the rotation. The operadic composition is defined almost exactly as for the operad $\mathcal{D}_k$. If we visualize the composition as gluing we must, however, rotate each disk before gluing it in. The operad $f\mathcal{D}_k$ is called the *framed little k-disks operad*. Both operads $f\mathcal{D}_k$ and $\mathcal{D}_k$ are pointed, with $f\mathcal{D}_k(0) = \mathcal{D}_k(0)$ consisting of the disk D^k with no hole in it.

To understand better the relation between the little disks operad and its framed version, we recall the following definitions introduced in [83].

Definition 4.1.1. *Let G be a topological group. A G-*operad *is an operad $\mathcal{P} = \{\mathcal{P}(n)\}_{n\geq 1}$ such that each $\mathcal{P}(n)$ is a left G-space and the composition map γ satisfies*

$$g(\gamma(x; x_1, \ldots, x_n)) = \gamma(gx; x_1, \ldots, x_n), \tag{4.1}$$

for each $x \in \mathcal{P}(n)$, $x_i \in \mathcal{P}(m_i)$, $1 \leq i \leq n$, and $g \in G$.

The framed little k-disks operad $f\mathcal{D}_k$ is an example of an $O(k)$-operad, with the orthogonal group acting by rotating the 'big' disk with all its framed holes inside. As we shall see, $f\mathcal{D}_k$ is a semidirect product of the orthogonal group $O(k)$ with the little disks operad $\mathcal{D}_k$ in the sense we introduce below. The terminology was suggested by P. Salvatore.

Suppose we have an (ordinary) operad $\mathcal{P}$ such that each $\mathcal{P}(n)$ is a left G-space and such that the composition map satisfies, under the notation of (4.1),

$$g\gamma(x; x_1, \ldots, x_n) = \gamma(gx; gx_1, \ldots, gx_n). \tag{4.2}$$

Then the *semidirect product* $\mathcal{P} \rtimes G = \{(\mathcal{P} \rtimes G)(n)\}_{n\geq 1}$ is the collection with $(\mathcal{P} \rtimes G)(n) := \mathcal{P}(n) \times G^{\times n}$, with the diagonal left action of the group G, diagonal right action of the symmetric group Σ_n and the composition map γ_G defined as

$$\begin{aligned}\gamma_G((x, g_1, \ldots, g_n); (x_1, g^1_1, \ldots, g^1_{m_1}), \ldots, (x_n, g^n_1, \ldots, g^n_{m_n})) :=\\ (\gamma(x; g_1x_1, \ldots, g_nx_n), g_1g^1_1, \ldots, g_1g^1_{m_1}, \ldots, g_ng^n_1, \ldots, g_ng^n_{m_n}),\end{aligned}$$

for $g_1, \ldots, g_n,\ g^1_1, \ldots, g^1_{m_1}, \ldots, g^n_1, \ldots, g^n_{m_n} \in G$, $x \in \mathcal{P}(n)$, $x_i \in \mathcal{P}(m_i)$, $1 \leq i \leq m$.

It is easy to verify that the semidirect product $\mathcal{P} \rtimes G$ is a G-operad as in Definition 4.1.1. There is a natural inclusion $\iota_\mathcal{P} : \mathcal{P} \hookrightarrow \mathcal{P} \rtimes G$ of operads given by $\iota_\mathcal{P}(n)(p) := p \times e^{\times n}$, where e is the unit of G, $p \in \mathcal{P}(n)$ and $n \geq 1$. Observe that the projection $p_\mathcal{P} : \mathcal{P} \rtimes G \to \mathcal{P}$ on the first factor is not an operadic morphism.

An example of an operad satisfying (4.2) is the little disk operad $\mathcal{D}_k$ with the $O(k)$-action given by rotating the 'big' disk with all its holes. We leave the proof of the following statement as an easy exercise to the reader.

Proposition 4.1.2. *There is a natural isomorphism of $O(k)$-operads*

$$\mathcal{D}_k \rtimes O(k) \cong f\mathcal{D}_k.$$

Let $\overset{\circ}{D}{}^k$ denote the interior of the standard disk D^k,

$$\overset{\circ}{D}{}^k := \{(x_1, \dots, x_k) \in \mathbb{R}^k;\ x_1^2 + \cdots + x_k^2 < 1\},$$

and let $\mathsf{F}(\overset{\circ}{D}{}^k, n)$ be the configuration space of n distinct numbered points in $\overset{\circ}{D}{}^k$, with the obvious right Σ_n-action. P. May proved, in [90, Theorem 4.8], that the nth piece $\mathcal{C}_k(n)$ of the little k-cubes operad has, for each $n \geq 1$, the Σ_n-equivariant homotopy type of $\mathsf{F}(\overset{\circ}{D}{}^k, n)$. Let us prove an analogous statement for the (framed) little k-disks operad.

Proposition 4.1.3. *For each $n \geq 1$, the space $f\mathcal{D}_k(n)$ is Σ_n-equivariantly homotopy equivalent to $\mathsf{F}(\overset{\circ}{D}{}^k, n) \times O(k)^{\times n}$.*

Similarly, the space $\mathcal{D}_k(n)$ has the Σ_n-equivariant homotopy type of $\mathsf{F}(\overset{\circ}{D}{}^k, n)$.

Proof. For a (framed) little disk $d : D^k \to \mathbb{R}^k$, denote by $c(d) \in D^k$ its *center*, $\mathrm{c}(d) := d(0)$. Let $d_* : T_0 D^k \to T_{\mathrm{c}(d)} D^k$ be the tangent map at zero and interpret it, using the canonical identifications $T_0 D^k \cong \mathbb{R}^k$ and $T_{\mathrm{c}(d)} D^k \cong \mathbb{R}^k$, as a map $d_* : \mathbb{R}^k \to \mathbb{R}^k$. There exists a unique element $\mathrm{ph}(d) \in O(k)$ such that, for some fixed scalar $\alpha > 0$,

$$d_*(t) = \alpha\ \mathrm{ph}(d)(t), \quad \text{for each } t \in \mathbb{R}^k.$$

We call $\mathrm{ph}(d)$ the *phase* of d. Define an equivariant map $a : f\mathcal{D}_k(n) \to \mathsf{F}(\overset{\circ}{D}{}^k, n) \times O(k)^{\times n}$ by

$$a(d_1, \dots, d_n) := (\mathrm{c}(d_1), \dots, \mathrm{c}(d_n)) \times (\mathrm{ph}(d_1), \dots, \mathrm{ph}(d_n)),$$

for $(d_1, \dots, d_n) \in f\mathcal{D}_k(n)$.

To define a homotopy inverse of a, we need the function $\rho : \mathsf{F}(\overset{\circ}{D}{}^k, n) \to \mathbb{R}_{>0}$ defined as

$$\rho(c_1, \dots, c_n) := \min \begin{cases} \frac{1}{2}\mathrm{dist}(c_i, c_j), & 1 \leq i \neq j \leq n, \\ 1 - \mathrm{dist}(c_i, 0), & 1 \leq i \leq n, \end{cases}$$

where dist is the standard Euclidean distance in $\mathbb{R}^k$ and $(c_1, \dots, c_n) \in \mathsf{F}(\overset{\circ}{D}{}^k, n)$. The meaning of the above definition is that $\rho = \rho(c_1, \dots, c_n)$ is the largest number such that the open disks with centers $c_1, \dots, c_n$ and radii ρ are mutually disjoint and all are subsets of D^k.

Next, observe that for each $c \in \overset{\circ}{D}{}^k$ and $g \in O(k)$ there exists exactly one *isometry* $d(c, g) : D^k \to \mathbb{R}^k$ such that $\mathrm{c}(d(c, g)) = c$ and $\mathrm{ph}(d(c, g)) = g$. The map $b : \mathsf{F}(\overset{\circ}{D}{}^k, n) \times O(k)^{\times n} \to f\mathcal{D}_k(n)$ is defined by the formula

$$b((c_1, \dots, c_n) \times (g_1, \dots, g_n)) := (\rho d(c_1, g_1), \dots, \rho d(c_n, g_n)), \quad \text{where } \rho = \rho(c_1, \dots, c_n),$$

for $(c_1, \ldots, c_n) \in \mathsf{F}(\mathring{D}^k, n)$ and $(g_1, \ldots, g_n) \in O(k)^{\times n}$. Let us prove that b is a homotopy inverse of a.

It is clear that $a \circ b = \mathbb{1}$, so it is enough to prove that $b \circ a$ is homotopic to $\mathbb{1}$. Observe that $(b \circ a)(d_1, \ldots, d_n)$ is obtained from $(d_1, \ldots, d_n) \in f\mathcal{D}_k(n)$ simply by rescaling the disks so that they all have radius $\rho(\mathrm{c}(d_1), \ldots, \mathrm{c}(d_n))$. Thus the most naive interpolation

$$h_t := (1-t)(a \circ b) + t\mathbb{1},\ 0 \leq t \leq 1, \tag{4.3}$$

is indeed a homotopy between $b \circ a$ and $\mathbb{1}$. It is clear that all constructions above are in fact Σ_n-equivariant, giving rise to an equivariant homotopy equivalence.

The second part of the statement is just the restriction to the unframed case. ∎

Let us clarify the relation between the little disks and little cubes operads. There is no obvious operadic morphism between them. To compare them, we must introduce an intermediate operad, the suboperad $t\mathcal{C}_k$ of the operad $\mathcal{C}_k$ consisting of 'true' k-cubes, i.e. little k-cubes for which, in the notation of **??**, the difference $y_i - x_i$ is the same for all $1 \leq i \leq n$.

Let $u : \mathcal{D}_k \to t\mathcal{C}_k$ be the map that assigns to each little k-disk the biggest 'true' little k-cube contained in it. Similarly, there is a map $v : t\mathcal{C}_k \to \mathcal{D}_k$ which gives to each 'true' little k-cube the biggest little k-disk contained in it. Both u and v are maps of operads and it is immediate to see that $v(n)$ is a Σ_n-equivariant homotopy inverse of $u(n)$, for each $n \geq 1$. The inclusion $\iota : t\mathcal{C}_k \to \mathcal{C}_k$ has a left inverse $j : \mathcal{C}_k \to t\mathcal{C}_k$ which gives to each little k-cube the biggest 'true' little k-cube contained in it and having the same center. The map j is, however, not a map of operads, but, as it can be easily seen, $j(n)$ is a homotopy inverse of $\iota(n)$, for each $n \geq 1$. The situation is summarized by the following diagram of operad maps.

$$\begin{tikzcd} \mathcal{D}_k \arrow[r, bend left, "u"] & t\mathcal{C}_k \arrow[l, bend left, "v"] \arrow[r, "\iota"] & \mathcal{C}_k \end{tikzcd}$$

As an easy consequence of the above considerations we get the following proposition.

Proposition 4.1.4. *The composition $\iota \circ u : \mathcal{D}_k \to \mathcal{C}_k$ induces an isomorphism of differential graded operads $H_*(\mathcal{D}_k)$ and $H_*(\mathcal{C}_k)$.*

We already observed **??** that the homology operad $H_*(\mathcal{C})$ of the little cubes operad (with $k = 2$) describes Gerstenhaber algebras, i.e. algebras with a commutative associative multiplication $\cdot$ and with a Lie bracket $[-,-]$ which are related by a distributivity law, see **??** for details. By Proposition 4.1.4, the same is true also for the operad $\mathcal{D} := \mathcal{D}_2$. The homology of the framed little disks operad $f\mathcal{D}$ describes another important class of algebras, called Batalin-Vilkovisky algebras.

Definition 4.1.5. *A* Batalin-Vilkovisky algebra *is a Gerstenhaber algebra* $(V, \cdot, [-,-])$ *together with a degree 1 linear map* $\Delta : V \to V$ *such that* $\Delta^2 = 0$ *and*

$$[a,b] = \Delta(ab) - (\Delta(a)b + (-1)^a a\Delta(b)), \tag{4.4}$$

for arbitrary $a, b, c \in V$.

Observe that Δ in the above definition is a differential, but not a derivation with respect to the commutative product, its deviation from being a derivation measured by the Lie bracket as expressed by (4.4). The following characterization of Batalin-Vilkovisky algebras was given in [34].

Theorem 4.1.6. *Batalin-Vilkovisky algebras are algebras over the operad* $H_*(f\mathcal{D})$.

The space $f\mathcal{D}(1)$ has the homotopy type of the circle S^1, so $H_*(f\mathcal{D}(1))$ has two generators, one of degree 0 corresponding to the identity $1 \in H_*(f\mathcal{D}(1))$, and one of degree $+1$ corresponding to the operation Δ.

Riemann spheres with punctures. The above examples live in real differential geometry. While there is no obvious complex analog of the unframed little disks operad $\mathcal{D}_k$, the complex analog of the framed little disks operad $f\mathcal{D} := f\mathcal{D}_2$ is the following.

Let Σ be a Riemann sphere, that is, a nonsingular complex projective curve of genus 0. By a *puncture* we mean a point of Σ together with a holomorphic embedding of the standard disk $U = \{z \in \mathbb{C};\ |z| \leq 1\}$ to Σ centered at the point. Thus a puncture is a point $p \in \Sigma$ with a holomorphic embedding $u : \tilde{U} \to \Sigma$, where $\tilde{U} \subset \mathbb{C}$ is an open neighborhood of U and $u(0) = p$. We say that two punctures $u_1 : \tilde{U}_1 \to \Sigma$ and $u_2 : \tilde{U}_2 \to \Sigma$ are *disjoint*, if $u_1(\overset{\circ}{U}) \cap u_2(\overset{\circ}{U}) = \emptyset$, where $\overset{\circ}{U} := \{z \in \mathbb{C};\ |z| < 1\}$ is the interior of U.

Let $\widehat{\mathcal{M}}_0(n)$ be the moduli space of Riemann spheres Σ with $n+1$ disjoint punctures $u_i : \tilde{U}_i \to \Sigma$, $0 \leq i \leq n$. The collection $\widehat{\mathcal{M}}_0 = \{\widehat{\mathcal{M}}_0(n)\}_{n\geq 1}$ forms an operad under sewing Riemannian spheres at punctures.

Let us describe this operadic structure using the $\circ_i$-formalism. Thus, let Σ represents an element $x \in \widehat{\mathcal{M}}_0(m)$ and Δ represents an element $y \in \widehat{\mathcal{M}}_0(n)$. For $1 \leq i \leq m$, let $u_i : \tilde{U}_i \to \Sigma$ be the ith puncture of Σ and let $u_0 : \tilde{U}_0 \to \Delta$ be the 0th puncture of Δ.

There certainly exists some $0 < r < 1$ such that both $\tilde{U}_0$ and $\tilde{U}_i$ contain the disk $U_{1/r} := \{z \in \mathbb{C};\ |z| < 1/r\}$. Let now $\Sigma_r := \Sigma - u_i(U_r)$ and $\Delta_r := \Delta - u_0(U_r)$. Define finally

$$\Xi := (\Sigma_r \bigsqcup \Delta_r)/\sim,$$

where the relation $\sim$ is given by

$$\Sigma_r \ni u_i(\xi) \sim u_0(1/\xi) \in \Delta_r,$$

for $r < |\xi| < 1/r$. It is immediate to see that Ξ is a well-defined punctured Riemannian sphere, with $n + m - 1$ punctures induced in the obvious manner from those of Σ and Δ, and that the class of the punctured surface Ξ in the moduli space $\widehat{\mathcal{M}}_0(m + n - 1)$ does not depend on the representatives Σ, Δ and on r. We define $x \circ_i y$ to be the class of Ξ. The symmetric group acts by permuting the labels of little disks.

Before describing the unit $1 \in \widehat{\mathcal{M}}_0(1)$, we introduce the following notation which will be useful in the sequel. Let $\mathbb{CP}^1$ be the complex projective space with homogeneous coordinates $[z, w]$ [134]. Let $0 := [0, 1] \in \mathbb{CP}^1$ and $\infty := [1, 0] \in \mathbb{CP}^1$. Recall that we have two canonical isomorphisms $p_\infty : \mathbb{CP}^1 - \infty \to \mathbb{C}$ and $p_0 : \mathbb{CP}^1 - 0 \to \mathbb{C}$ given by

$$p_\infty([z, w]) := z/w \text{ and } p_0([z, w]) := w/z.$$

Then $p_\infty^{-1} : \mathbb{C} \to \mathbb{CP}^1$ (resp. $p_0^{-1} : \mathbb{C} \to \mathbb{CP}^1$) is a puncture at 0 (resp. at ∞). We then define $1 \in \widehat{\mathcal{M}}_0(1)$ to be the class of $(\mathbb{CP}^1, p_0^{-1}, p_\infty^{-1})$. Notice that $\widehat{\mathcal{M}}_0$ is in fact a pointed operad, with $* \in \widehat{\mathcal{M}}_0(0)$ the class of $(\mathbb{CP}^1, p_0^{-1})$.

Proposition 4.1.7. *There exists a natural map of topological operads $j : f\mathcal{D} \to \widehat{\mathcal{M}}_0$.*

Proof. Let us describe, following [58], the construction of the map j. We start with the observation that elements $f \in f\mathcal{D}(n)$ can be interpreted as complex linear maps from the disjoint union of n copies of the standard complex unit disk U to U. Given such an element f, the composition $p_\infty^{-1} \circ f$ gives $\mathbb{CP}^1$ with n embedded holomorphic disks numbered by $\{1, \ldots, n\}$. In addition, we define the 0th disk to be p_0^{-1}. This gives a point in $\widehat{\mathcal{M}}_0(n)$ and completes the construction of the map $j(n)$. It is clear that the map $j(n)$ is well defined and a moment's reflections shows that it induces an operadic homomorphism. ■

Let $C_*(\widehat{\mathcal{M}}_0)$ be the differential graded operad of singular chains on $\widehat{\mathcal{M}}_0$. Recall **??** that the string background induces an action of the operad $C_*(\widehat{\mathcal{M}}_0)$ on the BRST complex $(\mathcal{H}, Q)$. The following theorem was proved in [58].

Theorem 4.1.8. *The absolute BRST homology $H_*(\mathcal{H}, Q)$ admits a natural structure of a Batalin-Vilkovisky algebra.*

Proof. The proof is obvious. The $C_*(\widehat{\mathcal{M}}_0)$-action on the BRST complex $(\mathcal{H}, Q)$ induces a $H_*(\widehat{\mathcal{M}}_0)$-action on its homology $H_*(\mathcal{H}, Q)$. This, composed with the map

$$H_*(j) : H_*(f\mathcal{D}) \to H_*(\widehat{\mathcal{M}}_0),$$

where $j : f\mathcal{D} \to \widehat{\mathcal{M}}_0$ is the map of Proposition 4.1.7, gives the advertised Batalin-Vilkovisky structure. ■

Little disks modules. The rest of this section has a speculative character. It is supposed to serve as an introduction to the compactification of configuration spaces in Section **??**.

One may read Theorem 4.1.3 as saying that the collection of configuration spaces of distinct points in the open disk $\overset{\circ}{D}{}^k$ has the equivariant homotopy type of an operad. We are going to study what happens when we consider configurations of points in a *general* smooth k-dimensional manifold M. Roughly speaking, we show that the collection of framed configuration spaces $f\mathsf{F}(M) = \{f\mathsf{F}(M,n)\}_{n\geq 1}$ (to be introduced in Definition 4.1.9) has the homotopy type of a specific right module over the framed little k-disks operad $f\mathcal{D}_k$. We formulate also an 'unframed' version of the above statement for parallelizable M.

Recall **??** that a topological collection $N = \{N(n)\}_{n\geq 1}$ is a right module over a topological operad $\mathcal{P}$ if we are given operations

$$\nu : N(n) \times \mathcal{P}(m_1) \times \cdots \times \mathcal{P}(m_n) \to N(m_1 + \cdots + m_n),$$

for $n \geq 1$, $m_i \geq 1$, $1 \leq i \leq n$, satisfying appropriate axioms, see **??** for details.

As in Theorem 4.1.3, we are going to compare the configuration space of points in M to a collection of little disks on M, so our first task will be to understand what a little disk in M should be.

The most general choice is that a little disk in M is just a smooth map (not even necessarily an embedding) $d : D^k \to M$. As before we denote by $\mathrm{c}(d) := d(0) \in M$ the center of d. We say that two little disks d_1, d_2 are disjoint if $d_1(\overset{\circ}{D}{}^k) \cap d_2(\overset{\circ}{D}{}^k) = \emptyset$. It will be clear soon that there is no way to fix the 'phase' of the disk unless we assume that M is parallelizable, so all disks are 'framed' from the beginning.

Let us denote by $f\mathcal{D}M = \{f\mathcal{D}M(n)\}_{n\geq 1}$ the collection of numbered little disks in M, with the obvious right action of the symmetric group. The collection $f\mathcal{D}M = \{f\mathcal{D}M(n)\}_{n\geq 1}$ has a right module structure over the framed little k-disks operad $f\mathcal{D}_k$ given as follows (compare also **??**).

Let $d : \bigsqcup_{s=1}^n D^k_s \to M$ be an element of $f\mathcal{D}M(n)$ and $d_s : \bigsqcup_{j=1}^{m_s} D^k_{j,s} \to D^k$ (where D^k_s and $D^k_{j,s}$ are identical copies of the standard k-disk) be elements of $f\mathcal{D}_k(m_s)$, $1 \leq s \leq n$. Then $\nu(d; d_1, \ldots, d_n) \in \mathcal{C}_k(m_1 + \cdots + m_n)$ is the map

$$\nu(d; d_1, \ldots, d_n) : \bigsqcup_{s=1}^{n} \bigsqcup_{j=1}^{m_s} D^k_{j,s} \longrightarrow M \tag{4.5}$$

given by

$$\nu(d; d_1, \ldots, d_n)|_{D^k_{j,s}} := d \circ d_s,\ 1 \leq j \leq m_s,\ 1 \leq s \leq n,$$

where we interpret d_s as a map $d_s : \bigsqcup_{j=1}^{m_s} D^k_{j,s} \to D^k_s$.

This gives a nice example of a right $f\mathcal{D}_k$-module, but if we do not put appropriate restrictions on little disks and their configurations, this example is very dull. Indeed, if we allow little disks to be all smooth maps and if we do not require them to be disjoint, then $f\mathcal{D}M(n)$ has Σ_n-homotopy type of the cartesian product $M^{\times n}$. To see this, define $a : f\mathcal{D}M(n) \to M^{\times n}$ by $a(d_1,\ldots,d_n) := (\mathrm{c}(d_1),\ldots,\mathrm{c}(d_n))$, for $(d_1,\ldots,d_n) \in f\mathcal{D}M(n)$. There is also a map $b : M^{\times n} \to f\mathcal{D}M(n)$ defined by $b(x_1,\ldots,x_n) := (d_{x_1},\ldots,d_{x_n})$ for $(x_1,\ldots,x_n) \in M^{\times n}$, where d_{x_i} denotes the constant little disk with $\mathrm{Im}(d_i) = \{x_i\}$, $1 \leq i \leq n$. It is immediate to see that a and b are Σ_n-equivariant maps, equivariantly homotopy inverse to each other.

Observe that the collection $\{M^{\times n}\}_{n\geq 1}$ is also a right $f\mathcal{D}_k$-module, with the structure given rather trivially by

$$\nu((x_1,\ldots,x_n);f_1,\ldots,f_n) := (\underbrace{x_1,\ldots,x_1}_{m_1 \text{ times}},\ldots,\underbrace{x_n,\ldots,x_n}_{m_n \text{ times}}),$$

for $(x_1,\ldots,x_n) \in M^{\times n}$ and $f_i \in f\mathcal{D}(m_n)$, $1 \leq i \leq n$.

This is certainly not what we want, because we secretly think of $f\mathcal{D}M$ as of an approximation of the configuration space. A better choice that makes $f\mathcal{D}M$ less trivial is to assume that the disks are disjoint, which means that one must also assume that each little disk is an embedding, otherwise the composition (4.5) need not lead to disjoint disks.

Under this choice, we have a map $a : f\mathcal{D}M(n) \to \mathsf{F}(M,n)$ sending each $(d_1,\ldots,d_n) \in f\mathcal{D}M(n)$ to $(\mathrm{c}(d_1),\ldots,\mathrm{c}(d_n)) \in \mathsf{F}(M,n)$. This a is still far from being a homotopy equivalence, since each little disk could be rotated by an element of $O(k)$ without changing its center, therefore the preimage $a^{-1}(x_1,\ldots,x_n)$ of each $(x_1,\ldots,x_n) \in M^{\times n}$ contains a highly nontrivial space $O(k)^{\times n}$. This can be improved by considering the framed configuration space.

Let $Gl(M)$ denote the principal $Gl(k)$-bundle of linear frames on M. We may always assume that M is Riemannian, so $Gl(M)$ reduces to orthogonal frames $O(M)$; denote by $\pi_M : O(M) \to M$ the bundle projection. There is a retraction $r_M : Gl(M) \to O(M)$ commuting with the bundle projections.

Let us introduce the *frame* of a little disk $d : D^k \to M$ as follows. Let $\{e_1,\ldots,e_k\}$ be the canonical basis of the tangent space of $\mathbb{R}^k$ at 0,

$$e_i := (0,\ldots,0,1,0,\ldots,0) \quad (1 \text{ at the } i\text{-th position}),\ 1 \leq i \leq k.$$

Let $\mathfrak{e} := (e_1,\ldots,e_k)$ be the corresponding orthogonal frame at $0 \in \mathbb{R}^k$. Let $d_* : T_0\mathbb{R}^k \to T_{\mathrm{c}(d)}M$ be the tangent map to d at $0 \in \mathbb{R}^k$. Then $d_*(\mathfrak{e})$ is an element of the frame bundle $Gl(M)$ and we define the frame of d to be

$$\mathrm{fr}(d) := r_M(d_*(\mathfrak{e})) \in \pi_M^{-1}(\mathrm{c}(d)).$$

Definition 4.1.9. *The* framed configuration space $f\mathsf{F}(M,n)$ *of n distinct numbered points in M is the space of all $(x_1,\mathfrak{f}_1;\ldots;x_n,\mathfrak{f}_n)$ where $(x_1,\ldots,x_n)\in\mathsf{F}(M,n)$ and $\mathfrak{f}_i\in\pi_M^{-1}(x_i)$, $1\le i\le n$.*

Observe that the obvious projection $p : f\mathsf{F}(M,n)\to\mathsf{F}(M,n)$ is a principal $O(k)^{\times n}$-bundle, in fact, $f\mathsf{F}(M,n)$ is the pullback of the diagram:

$$\begin{tikzcd} & O(M)^{\times n} \arrow[d, "\pi_M^{\times n}"] \\ \mathsf{F}(M,n) \arrow[r, "\text{inclusion}"] & M^{\times n} \end{tikzcd}$$

If M is parallelizable, then $f\mathsf{F}(M,n)$ is (noncanonically) isomorphic to $\mathsf{F}(M,n)\times O^{\times n}(k)$. For instance, the space $\mathsf{F}(\overset{\circ}{D}{}^k,n)\times O(k)^{\times n}$ of Theorem 4.1.3 is the framed configuration space $f\mathsf{F}(\overset{\circ}{D}{}^k,n)$ in disguise.

Let us define a Σ_n-equivariant map $a : f\mathcal{D}M(n)\to f\mathsf{F}(M,n)$ by

$$a(d_1,\ldots,d_n) := (\mathrm{c}(d_1),\mathrm{fr}(d_1);\ldots;\mathrm{c}(d_n),\mathrm{fr}(d_n)),\ \text{ for } (d_1,\ldots,d_n)\in f\mathcal{D}M(n).$$

A candidate for a homotopy inverse of a can be constructed as follows. It is clear that the function $\rho_1 : \mathsf{F}(M,n)\to\mathbb{R}_{>0}$ defined by

$$\rho_1(x_1,\ldots,x_n) := \min\left\{\tfrac12\mathrm{dist}(x_i,x_j),\ 1\le i\ne j\le n\right\},\ (x_1,\ldots,x_n)\in\mathsf{F}(M,n),$$

where $\mathrm{dist}(-,-)$ is the Riemannian distance in M, is continuous.

Suppose that the manifold M is complete. Then we have, for each point $x\in M$, the exponential map $\exp_x : T_xM\to M$ from the tangent space T_xM at x to M. Let us denote, for $r>0$, by B_r the open ball in T_xM centered at 0 with radius r. The *injectivity radius* of M at x, denoted $\mathfrak{i}(x)$, is the supremum of all $r>0$ such that $\exp_x$ is an embedding of B_r to M. The following theorem can be found in [8, page 241].

Theorem 4.1.10. *If the manifold M is complete, then the injectivity radius $\mathfrak{i} : M\to(0,\infty)$ is a continuous function.*

Thus, for a complete Riemannian manifold M, we have another continuous function $\rho_2 : \mathsf{F}(M,n)\to\mathbb{R}_{>0}$ defined by

$$\rho_2(x_1,\ldots,x_n) := \min\left\{\mathfrak{i}(x_i);\ 1\le i\le n\right\},\ \text{ for } (x_1,\ldots,x_n)\in\mathsf{F}(M,n).$$

We finally put

$$\rho := \min(\rho_1,\rho_2) : \mathsf{F}(M,n)\to\mathbb{R}_{>0}.$$

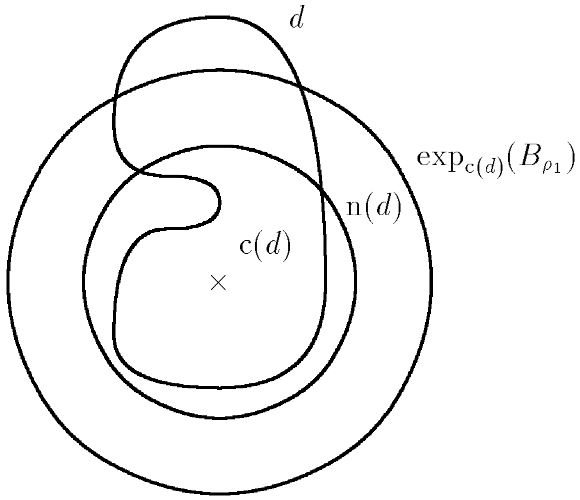


Figure 4.1: Little disk in M and its normalization. The amoeba is the image of d, the small disk is the image of $\mathrm{n}(d)$ and the big disk is the image $\exp_{\mathrm{c}(d)}(B_{\rho_1})$. The cross $\times$ indicates the centre $\mathrm{c}(d)$.

Let $(x_1, \mathfrak{f}_1; \ldots; x_n, \mathfrak{f}_n) \in f\mathsf{F}(M, n)$. Introduce an equivariant map $b : f\mathsf{F}(M, n) \to f\mathcal{D}M(n)$ by $b(x_1, \mathfrak{f}_1; \ldots; x_n, \mathfrak{f}_n) := (d_1, \ldots, d_n)$, with $d_i : D^k \to M$ defined as follows. There exists, for each $1 \leq i \leq n$, a unique linear isometric embedding $\xi_i : D^k \hookrightarrow T_{x_i}M$ such that the composition

$$\omega_i : D^k \xrightarrow{\xi_i} T_{x_i}M \xrightarrow{\exp_{x_i}} M$$

has the property that $\mathrm{fr}(\omega_i) = \mathfrak{f}_i$. Then

$$d_i(z) := \omega_i(\rho z),\ z \in D^k,\ \rho = \rho(x_1, \ldots, x_n). \tag{4.6}$$

So, we have constructed equivariant maps $a : f\mathcal{D}M(n) \to f\mathsf{F}(M, n)$ and $b : f\mathsf{F}(M, n) \to f\mathcal{D}M(n)$ obviously satisfying $a \circ b = \mathbb{1}$. Let us see what are our chances to construct a homotopy between $b \circ a$ and $\mathbb{1}$.

To simplify our language, call little disks of the form (4.6) *normal*. By definition, each normal disk d is uniquely determined by its center $\mathrm{c}(d)$, its phase $\mathrm{ph}(d)$ and the value of ρ. Each little disk d has its *normalization* $\mathrm{n}(d)$, i.e. a normal disk $\mathrm{n}(d)$ with the same phase and center as d.

The map $b \circ a$ clearly replaces a little disk $d : D^k \to M$ by its normalization, $(b \circ a)(d) = \mathrm{n}(d)$. The situation is schematically depicted in Figure 5.24. Thus a homotopy from $b \circ a$ to $\mathbb{1}$ shall interpolate between d and its normalization $\mathrm{n}(d)$, but if the image of d runs off the set $\exp_{\mathrm{c}(d)}(B_{\rho_1})$ as it does in Figure 5.24, it is not clear how to construct such an interpolation.

This can be fixed by introducing, for any $\delta > 0$, the submodule $f\mathcal{D}_\delta M = \{f\mathcal{D}_\delta M(n)\}_{n \geq 1}$ of $f\mathcal{D}M$ by defining $f\mathcal{D}_\delta M(n)$ to be the subspace of $f\mathcal{D}M(n)$ consisting of all disks in M whose radius (in the obvious sense) is less than δ.

If δ is small enough, the images of little disks d from $f\mathcal{D}_\delta M(n)$ are subsets of $\exp_{\mathrm{c}(d)}(B_{\rho_1})$, so we may consider them, using the identification via $\exp_{\mathrm{c}(d)} : T_{\mathrm{c}(d)}M \to M$, as maps to $\mathbb{R}^k \cong T_{\mathrm{c}(d)}M$. Now it makes sense to form a linear interpolation

$$d_t := (1-t)\mathrm{n}(d) + td,\ 0 \leq t \leq 1.$$

There is, however, no reason to expect that the map d_t is injective for all t. This can be achieved by further restricting to those disks in $f\mathcal{D}_\delta M(n)$ which are *conformal linear* which means, by definition, that they are compositions

$$D^k \xrightarrow{\xi} T_{\mathrm{c}(d)} \xrightarrow{\exp_{\mathrm{c}(d)}} M,$$

with some conformal linear ξ. Observe that the classical little k-disks operad $\mathcal{D}_k$ consists of conformal linear disks in the sense of the above definition. We may combine the above observations as:

Theorem 4.1.11. *Suppose that there is a positive δ such that $\mathsf{i}(x) > \delta$ for each $x \in M$. Then the right Σ_n-space $f\mathcal{D}_\delta M(n)$ consisting of conformal linear little disks with radius less than δ is Σ_n-equivariantly homotopy equivalent to the framed configuration space $f\mathsf{F}(M,n)$.*

Since the injectivity radius $\mathsf{i} : M \to \mathbb{R}_{\geq 0}$ is a continuous function (Theorem 4.1.10), the number δ in Theorem 4.1.11 always exists if M is compact.

A moments reflection shows, however, that the collection of conformal linear disks $f\mathcal{D}_\delta M$ will not longer be a right $f\mathcal{D}_k$-module unless M is flat. On the other hand, each manifold is flat if we are short-sighted enough, that is, if δ is very small. To make this more precise, suppose for a moment that there exists a limit

$$f\overline{\mathsf{F}}(M,n) := \varinjlim_{\delta \to 0^+} f\mathcal{D}_\delta M(n).$$

Since $f\overline{\mathsf{F}}(M,n)$ is a myopic image of $f\mathcal{D}M$, the collection $f\overline{\mathsf{F}}(M) = \{f\overline{\mathsf{F}}(M,n)\}_{\geq 1}$ is a $f\mathcal{D}_k$-module and each $f\overline{\mathsf{F}}(M,n)$ has the homotopy type of $f\mathsf{F}(M,n)$. A precise construction of $f\overline{\mathsf{F}}(M,n)$ is provided by the compactification of Section **??**.

For parallelizable manifolds, the constructions above have their unframed versions. Let us choose a section $s : M \to O(M)$ of the frame bundle $O(M)$ and denote

$$\mathcal{D}M(n) := \Big\{(d_1,\ldots,d_n) \in f\mathcal{D}M(n),\ \mathrm{fr}(d_i) = s(\mathrm{c}(d_i)),\ 1 \leq i \leq n\Big\}.$$

Then all considerations above apply, that is, the myopic limit

$$\overline{\mathsf{F}}(M,n) := \varinjlim_{\delta \to 0^+} \mathcal{D}_\delta M(n)$$

of configurations of 'unframed' linear conformal little disks has homotopy type of the configuration space $\mathsf{F}(M,n)$ and it gives rise to a natural right $\mathcal{D}_k$-module $\overline{\mathsf{F}}(M) := \{\overline{\mathsf{F}}(M,n)\}_{n\geq 1}$.

4.2. Operad of the Deligne-Knudsen-Mumford compactification of moduli spaces.

Version of October 10, 2000,

In this section we closely follow the exposition given by V. Ginzburg and M.M. Kapranov in [40]. Let $\mathcal{M}_{0,n+1}$ be the moduli space of $(n+1)$-tuples $(x_0,\ldots,x_n)$ of distinct numbered points on the complex projective line $\mathbb{CP}^1$ modulo projective automorphisms, that is, transformations of the form

$$\mathbb{CP}^1 \ni [\xi_1,\xi_2] \mapsto [a\xi_1 + b\xi_2, c\xi_1 + d\xi_2] \in \mathbb{CP}^1,$$

where $a,b,c,d \in \mathbb{C}$ with $ad-bc \neq 0$.

Choose a point $\infty \in \mathbb{CP}^1$ so that $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$. Since projective automorphisms act transitively and since those automorphism that fix ∞ are exactly those that restrict to affine automorphisms of $\mathbb{C}$, letting $x_0 = \infty$, one gets an isomorphism of $\mathcal{M}_{0,n+1}$ with the moduli space of n-tuples of distinct numbered points of $\mathbb{C}$ modulo affine automorphisms.

The moduli space $\mathcal{M}_{0,n+1}$ has a canonical compactification $\overline{\mathcal{M}}(n) \supset \mathcal{M}_{0,n+1}$ introduced by A. Grothendieck and F.F. Knudsen [21, 59]. The space $\overline{\mathcal{M}}(n)$ is the moduli space of stable $(n+1)$-pointed curves of genus 0:

Definition 4.2.1. *A* stable $(n+1)$-pointed curve of genus 0 *is an object* $(C;x_0,\ldots,x_n)$, *where* C *is a (possibly reducible) curve with at most nodal singularities and* $x_0,\ldots,x_n \in C$ *are distinct smooth points such that*

(i) each component of C *is isomorphic to* $\mathbb{CP}^1$,

(ii) the graph of intersections of components of C *(i.e., the graph whose vertices correspond to the components of* C *and edges to the intersection points of the components) is a tree, and*

(iii) each component of C *has at least three special points, where a special point means either one of the* x_i, $0 \leq i \leq n$, *or a singular point of* C *(the stability).*

Remark 4.2.2. It can be easily seen that a stable curve $(C;x_0,\ldots,x_n)$ admits no infinitesimal automorphisms that fix marked points $x_0,\ldots,x_n$, therefore $(C;x_0,\ldots,x_n)$ is 'stable' in the usual sense. Observe also that $\mathcal{M}_{0,n+1} = \emptyset$ for $n \leq 1$ and that $\mathcal{M}_{0,3}$ = the point corresponding to the 3-pointed stable curve $(\mathbb{CP}^1;\infty,1,0)$.

The space $\mathcal{M}_{0,n+1}$ forms an open dense part of $\overline{\mathcal{M}}(n)$ consisting of marked curves $(C; x_0, \ldots, x_n)$ such that C is isomorphic to $\mathbb{CP}^1$.

It follows from the results of [6, 29] that the space $\overline{\mathcal{M}}(n)$ is a smooth complex projective variety of dimension $n-2$. It has the following elementary construction described, for example, in [6, pages 64–65]. Let

$$\mathrm{Aff} := \{z \mapsto az + b;\ a, b \in \mathbb{C},\ a \neq 0\}$$

be the group of affine transformations of $\mathbb{C}$. The group Aff acts diagonally on $\mathbb{C}^n$ preserving the open subset

$$\mathbb{C}^n_* := \{(z_1, \ldots, z_n) \in \mathbb{C}^n;\ z_i \neq z_j \text{ for } 1 \leq i < j \leq n\}$$

of points with pairwise distinct coordinates. As noted above, one has an isomorphism

$$\mathcal{M}_{0,n+1} \cong \mathbb{C}^n_*/\mathrm{Aff}.$$

Let us denote by $\Delta \subset \mathbb{C}^n$ the principal diagonal, that is, the space

$$\Delta := \{(z, \ldots, z) \in \mathbb{C}^n;\ z \in \mathbb{C}\}.$$

There is an embedding $\iota_n : \mathcal{M}_{0,n+1} \hookrightarrow \mathbb{CP}^{n-2}$ defined as the composition

$$\mathcal{M}_{0,n+1} \cong \mathbb{C}^n_*/\mathrm{Aff} \subset (\mathbb{C}^n - \Delta)/\mathrm{Aff} \cong \mathbb{CP}^{n-2}.$$

where the isomorphism on the right is induced by

$$\mathbb{C}^n - \Delta \ni (z_1, \ldots, z_n) \longmapsto [z_1 - z_n, z_2 - z_n, \ldots, z_{n-1} - z_n] \in \mathbb{CP}^{n-2}. \tag{4.7}$$

The coordinate axes in $\mathbb{C}^n$ give, under correspondence (4.7), n distinguished points $p_1, \ldots, p_n \in \mathbb{CP}^{n-2}$. Let us blow up all the points p_i, $1 \leq i \leq n$, then blow up the proper transforms (= closures of the preimages of some open parts, see [45, page 165]) of the lines $<p_i, p_j>$, $1 \leq i < j \leq n$, then blow up the proper transforms of the planes $<p_i, p_j, p_k>$, $1 \leq i < j < k \leq n$, and so on. The resulting space is isomorphic to $\overline{\mathcal{M}}(n)$.

More formally, we define a tower of smooth varieties Y_k, $n \geq k \geq 3$:

$$\mathbb{CP}^{n-2} = Y_n \longleftarrow Y_{n-1} \longleftarrow Y_{n-2} \longleftarrow \cdots \longleftarrow Y_3 = \overline{\mathcal{M}}(n) \tag{4.8}$$

by downward induction on k as follows.

For a subset $I \subset [n] := \{1, \ldots, n\}$, let D_I denote the subvariety of $\mathbb{CP}^{n-2}$ corresponding, under (4.7), to the subset

$$\{(z_1, \ldots, z_n) \in \mathbb{C}^n;\ z_i = z_j \text{ for } i, j \in I\}.$$

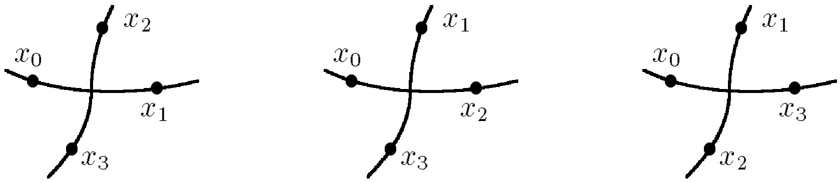


Figure 4.2: Three singular curves compactifying $\mathcal{M}_{0,4}$.

Observe that, in this notation, $D_{[n]-\{i\}} = \{p_i\}$, $D_{[n]-\{i,j\}} = \langle p_i, p_j \rangle$, &c. Assume by induction that the tower

$$\mathbb{CP}^{n-2} = Y_n \longleftarrow Y_{n-1} \longleftarrow Y_{n-2} \longleftarrow \cdots \longleftarrow Y_k$$

has already been constructed, for some k, $n \geq k > 3$, and denote by $\pi : Y_k \to \mathbb{CP}^{n-2}$ the composition. Let $\tilde{D}_I$ be the strict transform of D_I in Y_k. The following statement follows from [6, Lemma 3.2].

Lemma 4.2.3. *For each subset $I \subset [n]$ of cardinality $k-1$, $\tilde{D}_I$ is a smooth irreducible subvariety of Y_k. The intersection of any number of subvarieties $\tilde{D}_I$ with* $\mathrm{card}(I) = k-1$ *is either empty, or transverse.*

Now, Y_{k-1} is constructed by blowing up all subvarieties $\tilde{D}_I$ with $\mathrm{card}(I) = k-1$, in any order. By the transversality property of Lemma 4.2.3 the result does not depend on the order. This completes the induction step in the inductive construction of (4.8).

Example 4.2.4. For $n = 3$, the image of $\mathcal{M}_{0,4}$ in $\mathbb{CP}^1$ under the inclusion ι_3 is $\mathbb{CP}^1$ with three points $p_1 = \infty$, $p_2 = 0$ and $p_3 = 1$ removed. Tower (4.8) is trivial, so $\overline{\mathcal{M}}(3) = \mathbb{CP}^1$ is obtained from $\mathcal{M}_{0,4}$ by adding three points corresponding to three singular stable 4-curves symbolized in Figure 4.2.

Example 4.2.5. Let us discuss the case $n = 4$. The image of the embedding $\iota_4 : \mathcal{M}_{0,5} \hookrightarrow \mathbb{CP}^2$ is the complement of the configuration of six lines $\langle p_i, p_j \rangle$, $1 \leq i < j \leq 4$ as in Figure 4.3. The line $\langle p_i, p_j \rangle$ corresponds to the degenerate configuration $z_k = z_l$, $\{i,j,k,l\} = \{1,2,3,4\}$. The lines intersect at four triple points p_i, $1 \leq i \leq 4$, corresponding to the degenerate configurations $z_j = z_k = z_l$, where again $\{i,j,k,l\} = \{1,2,3,4\}$.

The tower $\mathbb{CP}^2 \longleftarrow Y_3 = \overline{\mathcal{M}}(4)$ is given by blowing up the four points p_i, $1 \leq i \leq 4$. The complement of $\mathcal{M}_{0,5}$ in $\overline{\mathcal{M}}(4)$ consists of strict transforms $\tilde{\ell}_{ij}$ of the lines $\langle p_i, p_j \rangle$, $1 \leq i < j \leq 4$, and four exceptional lines $\tilde{\ell}_i$, $1 \leq i \leq 4$, added by the blow up. In the rest of this example, i, j, k and l will be natural numbers such that $\{i,j,k,l\} = \{1,2,3,4\}$.

Each $\tilde{\ell}_{ij}$ is a smooth curve containing three special points – the point $\tilde{p}_{(ij)}$ (resp. $\tilde{p}_{(ji)}$) at which ℓ_{ij} transversally meets $\tilde{\ell}_i$ (resp $\tilde{\ell}_j$) and the point $\tilde{q}_{ij} = \tilde{q}_{kl}$ at which $\tilde{\ell}_{ij}$ transversally

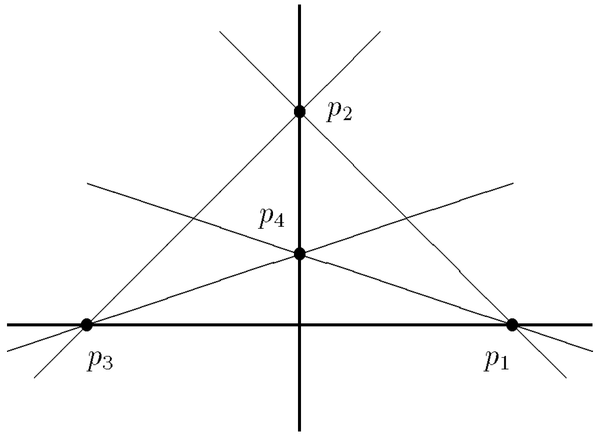
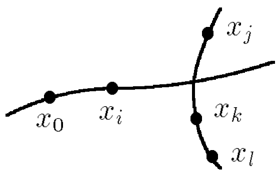


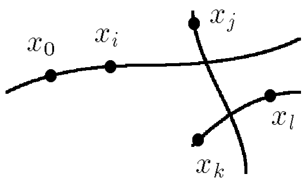
Figure 4.3: A configuration of six lines in $\mathbb{CP}^2$.

intersects $\tilde{\ell}_{kl}$. Thus the compactification $\overline{\mathcal{M}}(4)$ adds to $\mathcal{M}_{0,5}$ the following types of singular pointed curves

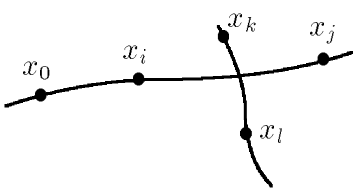
Type I. Curves corresponding to points of $\tilde{\ell}_i$ but different from $\tilde{p}_{(ij)}$:



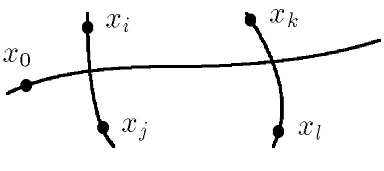
Type II. Curves corresponding to $\tilde{p}_{(ij)}$:



Type III. Curves corresponding to points of $\tilde{\ell}_{ij}$ but different from $\tilde{p}_{(ij)}$, $\tilde{p}_{(ji)}$ and $\tilde{q}_{ij}$:



Type IV. Curves corresponding to $\tilde{q}_{ij} = \tilde{q}_{kl}$:



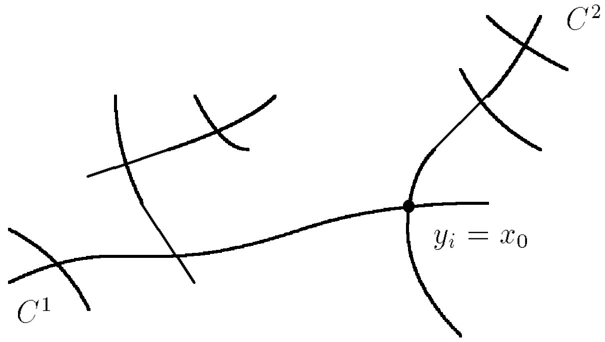


Figure 4.4: The pseudo-operadic composition of $\overline{\mathcal{M}} = \{\overline{\mathcal{M}}(n)\}_{n\geq 2}$.

Pseudo-operad structure. The family of spaces $\overline{\mathcal{M}} = \{\overline{\mathcal{M}}(n)\}_{n\geq 2}$ forms a topological pseudo-operad, with structure operations given as follows. The composition map

$$\circ_i : \overline{\mathcal{M}}(k) \times \overline{\mathcal{M}}(l) \longrightarrow \overline{\mathcal{M}}(k+l-1) \tag{4.9}$$

is, for $k, l \geq 2$, $1 \leq i \leq k$, defined by

$$(C^1; y_0, \dots, y_k) \times (C^2; x_0, \dots, x_l) \longmapsto (C; y_0, \dots, y_{i-1}, x_0, \dots, x_l, y_{i+1}, \dots, y_k)$$

where C is the curve obtained from the disjoint union $C^1 \coprod C^2$ by identifying x_0 with y_i, introducing a nodal singularity. The composition is illustrated in Figure 4.4. The symmetric group acts on $\overline{\mathcal{M}}(n)$ by

$$(C, x_0, x_1, \dots, x_n) \longmapsto (C, x_0, x_{\sigma(1)}, \dots, x_{\sigma(n)}),\ \sigma \in \Sigma_n.$$

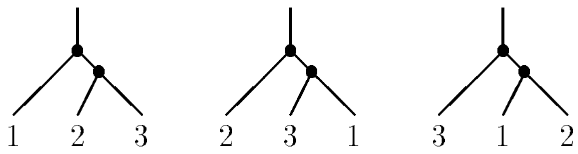
We call $\overline{\mathcal{M}} = \{\overline{\mathcal{M}}(n)\}_{n\geq 2}$ the *configuration pseudo-operad*, since $\overline{\mathcal{M}}(n)$ can be regarded as (compactified) configuration space of points on $\mathbb{CP}^1$.

One can associate to a point $(C; x_0, \dots, x_n) \in \overline{\mathcal{M}}(n)$ a rooted n-tree

$$T = \mathit{Tree}(C; x_0, \dots, x_n) \in \mathtt{Tree}_n$$

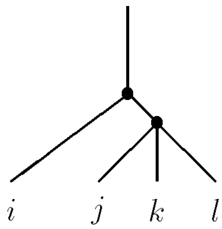
whose vertices correspond to irreducible components of C. The vertices corresponding to two components C_1, C_2 are joined by an (internal) edge if and only if $C_1 \cap C_2 \neq \emptyset$. A leaf labelled by i, $1 \leq i \leq n$, is attached to the vertex corresponding to the component containing x_i. The root is attached to the vertex corresponding to the component containing x_0. Property (iii) of stable curves ensures that $\mathit{Tree}(C; x_0, \dots, x_n)$ is a reduced tree thus, in fact, $\mathit{Tree}(C; x_0, \dots, x_n) \in \mathtt{Tree}^r_n$.

Example 4.2.6. Trees corresponding to singular curves in Figure 4.2 are

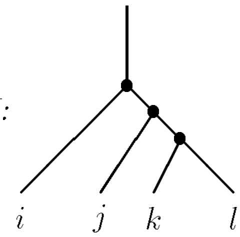


Trees corresponding to singular curves listed in Example 4.2.5 are:

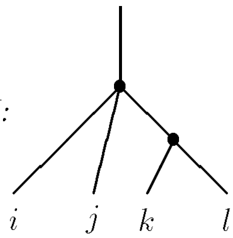
Type I:



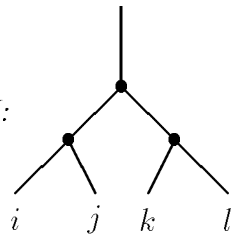
Type II:



Type III:



Type IV:



For any reduced n-tree T, let $\mathcal{M}(T) \subset \overline{\mathcal{M}}(n)$ be the subset consisting of stable curves $(C; x_0, \ldots, x_n)$ such that $\mathit{Tree}(C; x_0, \ldots, x_n) = T$. In this way we obtain an algebraic stratification

$$\overline{\mathcal{M}}(n) = \bigcup_{T \in \mathtt{Tree}_n^r} \mathcal{M}(T), \;\; n \geq 2.$$

This stratification has the following properties (see [6]):

$$\operatorname{codim}(\mathcal{M}(T)) = \operatorname{card}(\operatorname{Edg}(T)),$$

where $\operatorname{Edg}(T)$ is the set of internal edges of the tree T, and

$$\mathcal{M}(T) \subset \overline{\mathcal{M}(S)} \text{ if and only if } T \geq S.$$

Recall that $T \geq S$ means that S is obtained from T by collapsing one or more edges. In the above display, $\overline{\mathcal{M}(S)}$ denotes the closure of $\mathcal{M}(S)$ in $\overline{\mathcal{M}}(n)$.

In particular, 0-dimensional strata are labelled by binary trees. Codimension 1 strata correspond to trees with two vertices. Their closures are precisely the irreducible components of $\overline{\mathcal{M}}(n) - \mathcal{M}_{0,n+1}$, which is a normal crossing divisor. In addition, we have the following result.

Proposition 4.2.7. *For each $n \geq 3$ and $T \in \mathtt{Tree}_n^r$, there are canonical direct product decompositions*

$$\mathcal{M}(T) = \prod_{v \in \operatorname{Vert}(T)} \mathcal{M}_{0,|\operatorname{inp}(v)|+1},$$

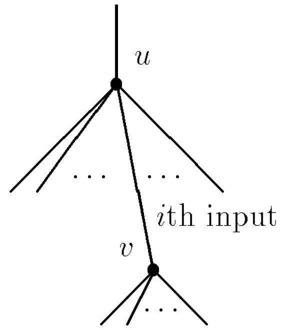


Figure 4.5: The tree $T_i(k,l)$. The vertex u has k inputs, the vertex v has l inputs.

and

$$\overline{\mathcal{M}(T)} = \prod_{v \in \mathrm{Vert}(T)} \mathcal{M}(|\mathrm{inp}(v)|).$$

In particular, the closure of each stratum is smooth.

Proof. The first equality means that a pointed curve $(C; x_0, \ldots, x_n) \in \mathcal{M}(T)$ is uniquely determined, up to isomorphism, by the projective equivalence classes of the configurations formed on each component by the marked points x_i and the double points which happen to lie on this component. This is obvious. The second equality follows from the first one once we note that

$$\overline{\mathcal{M}(T)} = \bigcup_{S \geq T} \mathcal{M}(S).$$

■

For $k, l \geq 2$ and $1 \leq i \leq k$, let $T_i(k,l)$ be the tree in Figure 4.5.

Corollary 4.2.8. *The structure maps $\circ_i$ of the pseudo-operad $\overline{\mathcal{M}} = \{\overline{\mathcal{M}}(n)\}_{n \geq 2}$ can be identified with the embedding of the closed stratum*

$$\overline{\mathcal{M}}(k) \times \overline{\mathcal{M}}(l) \cong \overline{\mathcal{M}(T_i(k,l))} \hookrightarrow \overline{\mathcal{M}}(k+l-1).$$

4.3. Compactification of configuration spaces of points in $\mathbb{R}^N$.

Version of May 26, 2000

The aim of this section is to construct a compactification of the moduli space $\mathring{\mathsf{F}}_k(n)$ of configurations of n distinct points in the k-dimensional Euclidean plane $\mathbb{R}^k$ modulo the action of the affine group, described by Getzler and Jones in [36] and denoted by $\mathsf{F}_k(n)$. The authors of [36] also stated that the collection $\mathsf{F}_k := \{\mathsf{F}_k(n)\}_{n\geq 1}$ has a natural structure of a topological operad. This was a well-known fact for $k = 1$, because the collection $\mathsf{F}_1 = \{\mathsf{F}_1(n)\}_{n\geq 1}$ is nothing else but the operad $K = \{K_n\}_{n\geq 1}$ of the associahedra **??**. The operad F_2 plays an important rôle in topological closed string field theory.

We are going to present a construction that stresses the operadic nature of the compactification. In [83] we observed that configuration spaces posses a kind of a partial operad structure, which we recall in the appendix to this section. This motivated us to establish a general theory of partial operads and to show that the compactification is a certain operadic completion of the partial operad of configuration spaces, making it manifestly an operad.

We are not going to repeat the construction of the operadic completion in its full generality here, we just show what it gives for the special cases of the partial operad related to the moduli space of configurations mentioned above, which we called in [83] the partial operad of *virtual configurations.*

Let us recall the definition of the moduli space $\mathring{\mathsf{F}}_k(n)$. As before, let $\mathsf{F}(\mathbb{R}^k, n)$ be the configuration space of n distinct labeled points in $\mathbb{R}^k$, $n \geq 1$.

Definition 4.3.1. *Let* $\mathrm{Aff}(\mathbb{R}^k)$ *be the affine group of translations and dilatations of* $\mathbb{R}^k$. *The group* $\mathrm{Aff}(\mathbb{R}^k)$ *acts in the obvious manner on* $\mathsf{F}(\mathbb{R}^k, n)$. *Define* $\mathring{\mathsf{F}}_k(n)$ *to be the quotient space*

$$\mathring{\mathsf{F}}_k(n) := \mathsf{F}(\mathbb{R}^k, n)/\mathrm{Aff}(\mathbb{R}^k),\ \ n, k \geq 1. \tag{4.10}$$

The moduli spaces of (5.36) form a collection $\mathring{\mathsf{F}}_k = \{\mathring{\mathsf{F}}_k(n)\}_{n\geq 1}$ with the symmetric group permuting the labels of the points. As usual, we can extend $\mathring{\mathsf{F}}_k$ to a functor on the category of finite sets; for a finite set X then $\mathring{\mathsf{F}}_k(X)$ is the moduli space of configurations of distinct points labeled by elements of X. It is immediate to see that each $\eta \in \mathring{\mathsf{F}}_k(X)$ has a unique representative $\{z^s \in \mathbb{R}^k\}_{s\in X}$ such that

$$\sum_{s\in X} z^s = 0 \text{ and } \sum_{s\in X} |z^s|^2 = 1, \tag{4.11}$$

where $|-|$ is the Euclidean norm. We call $\{z^s \in \mathbb{R}^k\}_{s\in X}$ satisfying (4.11) the *normal representative* of $\eta \in \mathring{\mathsf{F}}_k(X)$ and we call sequences satisfying (4.11) to be *normal*.

The above can be rephrased by saying that each sequence $\vec{w} = \{w^s \in \mathbb{R}^k\}_{s\in X}$ has a unique *normalization* $\mathrm{nor}(\vec{w})$ which is, by definition, a normal sequence $\vec{z} = \{z^s \in \mathbb{R}^k\}_{s\in X}$ congruent to $\vec{w}$ modulo the action of the affine group. The set of all normal sequences $\vec{z} = \{z^s \in \mathbb{R}^k\}_{s\in X}$ is clearly compact in $(\mathbb{R}^k)^X$.

Metric trees. In this section, by a metric tree we mean a rooted tree together with a 'length' function (or a metric) $h : \mathrm{edg}(T) \to \mathbb{R}_{\geq 0}$ from the set of (inner) edges of T into

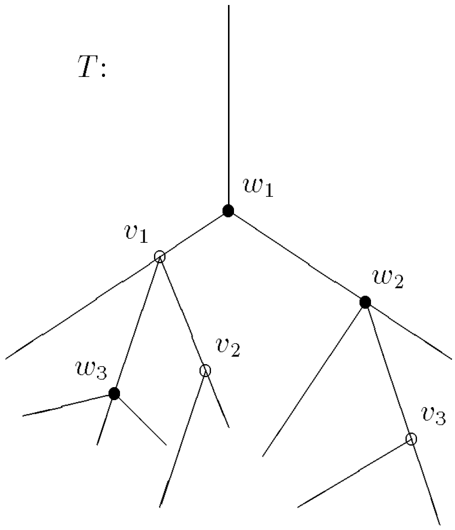


Figure 4.6: Suppose that $\mathfrak{Z}(h) = \{w_1, w_2, w_3\}$ (solid vertices). Then T_{w_1} is the subtree with $\mathrm{Vert}(T_{w_1}) = \{w_1, v_1, v_2\}$, T_{w_2} is the subtree with $\mathrm{Vert}(T_{w_2}) = \{w_2, v_3\}$ and T_{w_3} is the corolla with vertex w_3.

non-negative real numbers. So metric trees are the same as in ?? except that we admit their edges to have an arbitrary non-negative length. Let us denote by $\mathrm{Met}(T)$ the set of all metrics on a given tree T; we are sure that there is no danger to mistake it with the notation used in ??. Let us also denote by $\mathtt{MTree}^r_n$ the space of all reduced metric trees in the above sense, that is,

$$\mathtt{MTree}^r_n := \{(T, h);\ T \in \mathtt{Tree}^r_n,\ h \in \mathrm{Met}(T)\}.$$

For any $e \in \mathrm{edg}(T)$, let $v_e \in \mathrm{Vert}(T)$ be the unique vertex of T so that e is the output of v. Let $\mathrm{rv}(T)$ be the root vertex of T. Given $(T, h) \in \mathtt{MTree}^r_n$, define the 'zero set' of h to be a subset of $\mathrm{Vert}(T)$ defined by

$$\mathfrak{Z}(h) := \{v_e \in \mathrm{Vert}(T);\ h(e) = 0\} \cup \{\mathrm{rv}(T)\}. \tag{4.12}$$

The metric h also determines a decomposition of T into disjoint subtrees $\{T_w\}_{w \in \mathfrak{Z}(h)}$ as follows. For each $w \in \mathfrak{Z}(h)$, T_w is defined by

$$\mathrm{Vert}(T_w) := \{v \in \mathrm{Vert}(T);\ v \leq w \text{ and there is no } u \in \mathfrak{Z}(h) \text{ such that } v \leq u < w\},$$

where $<$ is the partial order on $\mathrm{Vert}(T)$ introduced in ??. The idea of the construction of T_w is illustrated in Figure 4.6.

Finally, let T_h denote the tree obtained from T by shrinking each subtree T_w, $w \in \mathfrak{Z}(T)$, to a vertex which we denote again by w. The construction of T_h from the metric tree (T, h) of Figure 4.6 is illustrated in Figure 4.7.

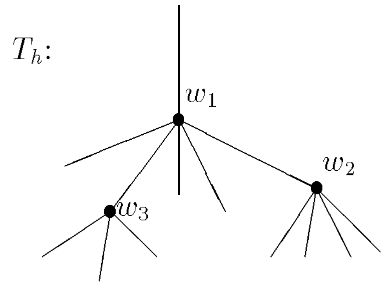


Figure 4.7: Construction of T_h from the metric tree (T, h).

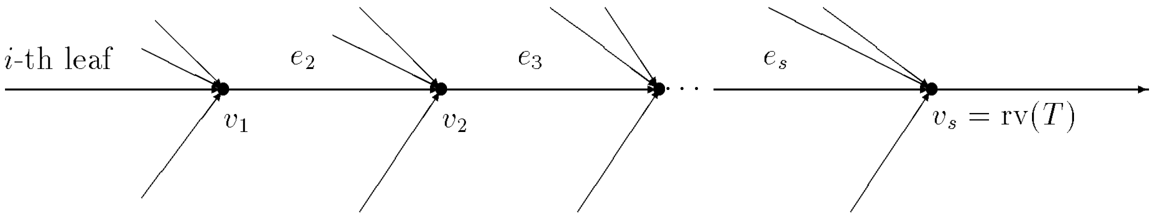


Figure 4.8: A path in the tree T.

The compactification. Let us come back to the collection $\mathring{\mathsf{F}}_k = \{\mathring{\mathsf{F}}_k(n)\}_{n\geq 1}$. For any reduced $T \in \mathtt{Tree}^r_n$ consider the space

$$Y[T] := \mathrm{Met}(T) \times \mathring{\mathsf{F}}_k(T). \tag{4.13}$$

Informally, elements of $Y[T]$ are metric trees with vertices labeled by the collection $\mathring{\mathsf{F}}_k$. More formally, elements of $Y[T]$ are sequences

$$\xi = \{(t_v, [\vec{z}_v]);\ v \in \mathrm{Vert}(T)\},\ (t_v, [\vec{z}_v]) \in \mathbb{R}_{\geq 0} \times \mathring{\mathsf{F}}_k(\mathrm{inp}(v)), \tag{4.14}$$

where $t_v := h(e_v)$, the length of the outcoming edge e_v of $v \in \mathrm{Vert}(T)$ if v is not a root, and $t_{\mathrm{rv}(T)} := 0$. By $\vec{z}_v$ we denote the normal reprezentative $\vec{z}_v = \{z^e_v \in \mathbb{R}^k\}_{e\in\mathrm{inp}(v)}$ of an element of $\mathring{\mathsf{F}}_v(\mathrm{inp}(v))$.

We define, for $T \in \mathtt{Tree}^r_n$, a map $\omega_T : Y[T] \to (\mathbb{R}^k)^n$ as follows. For any $1 \leq i \leq n$ there is in T a unique path from the i-th input leaf to the root, as in Figure 4.8. Using notation (4.14), we put

$$\omega_i(\xi) := z^{e_s}_{v_s} + t_{v_{s-1}} \cdot z^{e_{s-1}}_{v_{s-1}} + \cdots + t_{v_1} \cdots t_{v_{s-1}} \cdot z^i_{v_1} \in \mathbb{R}^k$$

and

$$\omega_T(\xi) := (\omega_1(\xi), \ldots, \omega_n(\xi)) \in (\mathbb{R}^k)^n. \tag{4.15}$$

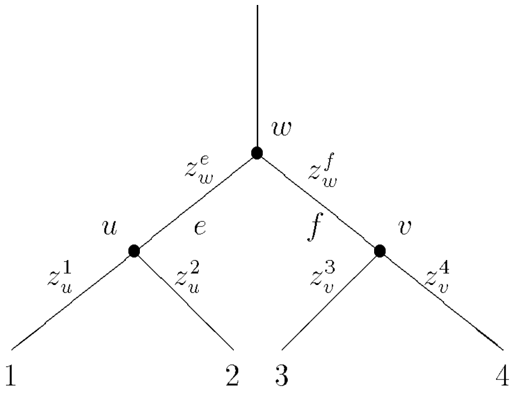


Figure 4.9: An $\mathring{\mathsf{F}}_1$-colored tree $T \in \mathtt{Tree}^r_4$.

Example 4.3.2. For $T = t(n)$ (the n-corolla), $n \geq 2$, the map

$$\mathring{\mathsf{F}}_k(n) \cong Y[t(n)] \xrightarrow{\omega_{t(n)}} \mathsf{F}(\mathbb{R}^k, n)$$

maps $\eta \in \mathring{\mathsf{F}}_k(n)$ to its unique normal representative in $\mathsf{F}(\mathbb{R}^k, n)$.

The map ω_T in fact reflects a partial operad structure on the configuration space. The 'partiality' means that $\omega_T(\xi)$ is an element of the configuration $\mathsf{F}(\mathbb{R}^k, n)$ space only for ξ belonging to some subset $U[T]$ of $Y[T]$,

$$U[T] := \{\xi \in Y[T];\ \text{all } \omega_i(\xi) \text{ are distinct, } 1 \leq i \leq n\}. \tag{4.16}$$

It is thus appropriate to call the subset $U[T]$ the set of *composable elements*. We used the square-bracket notation $U[T]$ to underline that $U[T]$ is not a set U(T) of U-colorings of T for some collection U.

Observe that, if $\xi = (h, \zeta) \in U[T] \subset \mathrm{Met}(T) \times \mathring{\mathsf{F}}_k(T)$, then $\mathfrak{Z}(h) = \{\mathrm{rv}(T)\}$. The opposite implication is not true for other trees than corollas, as illustrated in the following example.

Example 4.3.3. Let $k = 1$ and consider the tree T in Figure 4.9. The root w is colored by $[z^e_w, z^f_w] \in \mathring{\mathsf{F}}_1(\{e, f\})$ with $z^1_w := -\frac{1}{\sqrt{2}}$, $z^2_w := \frac{1}{\sqrt{2}}$. The colors of the remaining vertices are the same, that is,

$$[z^e_w, z^f_w] = [z^1_u, z^2_u] = [z^3_v, z^4_v].$$

This defines an element $\zeta \in \mathring{\mathsf{F}}_1(T)$. For two positive parameters $s, t \in \mathbb{R}_{\geq 0}$ define $h_{s,t} \in \mathrm{Met}(T)$ by $h_{s,t}(e) := s$, $h_{s,t}(f) := t$ and put $\xi_{s,t} := (h_{s,t}, \zeta) \in Y[T]$ with $\zeta \in \mathring{\mathsf{F}}_1(T)$ as above. An easy calculation shows that

$$\omega_T(\xi_{s,t}) = \frac{1}{\sqrt{2}}(-1 - s, -1 + s, 1 + t, 1 - t) \in \mathbb{R}^4.$$

From this we conclude that $\xi_{s,t} \in U[T]$ if and only if

$$s, t \neq 0, \ |s - t| \neq 2 \text{ and } |s + t| \neq 2.$$

Thus, for instance, the element $\xi_{1,1} = (h_{1,1}, \zeta) \in Y[T]$ does not belong to $U[T]$ though $\mathfrak{Z}(h_{1,1}) = \{\mathrm{rv}(T)\}$.

The following lemma, formulated in [83, page 192], will be useful in our proofs.

Lemma 4.3.4. *For any $n \geq 2$ and a reduced tree $T \in \mathtt{Tree}^r_n$, the composition*

$$\lambda_T : U[T] \xrightarrow{\omega_T} \mathsf{F}(\mathbb{R}^k, n) \xrightarrow{\text{proj.}} \mathring{\mathsf{F}}_k(n)$$

is a monomorphism.

We make the set $U[T]$ a bit bigger, allowing also some (h, ζ) such that $\mathfrak{Z}(h) \neq \{\mathrm{rv}(T)\}$, namely those whose restriction to each subtree T_w, $w \in \mathfrak{Z}(h)$, is composable. This can be formally done as follows.

Any $\xi = (h, \zeta) \in \mathrm{Met}(T) \times \mathring{\mathsf{F}}_k(T) = Y[T]$ determines, for each $w \in \mathfrak{Z}(T)$, the restriction

$$r_w(\xi) := (h|_{T_w}, \zeta|_{T_w}) \in Y[T_w]. \tag{4.17}$$

Define

$$\widehat{U}[T] := \{\xi \in Y[T];\ r_w(\xi) \in U(T_w) \text{ for all } w \in \mathfrak{Z}(h)\} \tag{4.18}$$

and call $\widehat{U}[T]$ the *extended* set of composables.

There are special elements of $Y[T]$ of the form $(0, \psi) \in \mathrm{Met}(T) \times \mathring{\mathsf{F}}_k(T)$, where 0 denotes the trivial metric assigning to each edge of T length zero. It is clear that each such element belongs to $\widehat{U}[T]$. We call such elements *primitive*. Assigning to each $\psi \in \mathring{\mathsf{F}}_k(T)$ its related primitive $(0, \psi)$ defines an inclusion

$$\iota_T : \mathring{\mathsf{F}}_k(T) \hookrightarrow \widehat{U}[T]. \tag{4.19}$$

The set $\widehat{U}[T]$ was defined in such a way that each $\xi = (h, \zeta) \in \widehat{U}[T]$ determines $\gamma(\xi) \in \mathring{\mathsf{F}}_k(T_h)$ by

$$\gamma(\xi) := \bigtimes_{w \in \mathrm{Vert}(T_h) = \mathfrak{Z}(h)} [\omega_{T_w}(r_w(\xi))]. \tag{4.20}$$

Observe that

$$\gamma(0, \psi) = \psi,$$

for each $\psi \in \mathring{\mathsf{F}}_k(T)$. This can be expressed as saying that "γ is the identity on primitives." The following lemma will be used in the proof of Theorem 4.3.11.

Lemma 4.3.5. *Let $\xi_1 = (h_1, \zeta_1)$ and $\xi_2 = (h_2, \zeta_2)$ be elements of $\widehat{U}[T]$ such that $\mathfrak{Z}(h_1) = \mathfrak{Z}(h_2)$. If we denote $S := T_{h_1} = T_{h_2}$, then*

$$\gamma(\xi_1) = \gamma(\xi_2) \text{ (equality of elements of } \overset{\circ}{\mathsf{F}}_k(S))$$

implies that $\xi_1 = \xi_2$.

Proof. By definition (4.20) of γ, it is enough to verify that $[\omega_{T_w}(r_w(\xi_1))] = [\omega_{T_w}(r_w(\xi_2))]$ implies that $r_w(\xi_1) = r_w(\xi_2)$, for any $w \in \mathfrak{Z}(h_1) = \mathfrak{Z}(h_2)$. This immediately follows from Lemma 4.3.4 applied on the trees T_w. ■

Define the reduced collection $\widehat{U} := \{\widehat{U}(n)\}_{n\geq 2}$ by

$$\widehat{U}(n) := \bigsqcup_{T \in \mathtt{Tree}_n^r} \widehat{U}[T] \text{ for } n \geq 2. \tag{4.21}$$

Elements of $\widehat{U}(n)$ are called *virtual configurations*. The compactification $\mathsf{F}_k(n)$ is, for $n \geq 2$, defined by

$$\mathsf{F}_k(n) := \widehat{U}(n)/\sim \tag{4.22}$$

where the relation $\sim$ is given by

$$\widehat{U}(n) \supset \widehat{U}[T] \ni \xi \sim (0, \gamma(\xi)) \in \widehat{U}(T_h) \subset \widehat{U}(n), \tag{4.23}$$

for $T \in \mathtt{Tree}_n^r$. We put $\mathsf{F}_k(1) := *$.

Lemma 4.3.6. *The projection $\widehat{U}(n) \xrightarrow{\text{proj.}} \mathsf{F}_k(n)$ is, for any $n \geq 1$, an open map.*

Proof. This is the first of many statements in this section whose 'formal' proof would be, due to the lack of an effective notation, very difficult to give, but whose validity is 'evident.'

By [56, Theorem 10, page 97], the projection from $\widehat{U}(n)$ to the quotient-space $\widehat{U}(n)/\sim$ is open if an only if for each open $U \subset \widehat{U}(n)$ is open also the set

$$\{\xi \in \widehat{U}(n);\ \xi \sim u \text{ for some } u \in U\}.$$

The openness of this set follows from the fact that the relation $\sim$ is defined with the help of the maps γ introduced in (4.20) which are continuous (this is clear) and *open*, which can be seen easily. ■

Proposition 4.3.7. *The collection $\mathsf{F}_k = \{\mathsf{F}_k(n)\}_{n\geq 1}$ has a natural structure γ_{F} of a topological operad. The natural map*

$$\rho : \mathcal{F}(\overset{\circ}{\mathsf{F}}_k) \to \mathsf{F}_k, \tag{4.24}$$

where $\mathcal{F}(\overset{\circ}{\mathsf{F}}_k)$ is the free operad on the collection $\overset{\circ}{\mathsf{F}}_k$, is an isomorphism of `Set`*-operads.*

Proof. The operad structure on F_k is given as follows. For $n \geq 2$ define

$$Y(n) := \bigsqcup_{T \in \mathtt{Tree}_n^r} Y[T].$$

The collection $Y = \{Y(n)\}_{n\geq 2}$ is a topological pseudo-operad, with the composition defined by grafting as in **??**. It is clear that the reduced collection $\widehat{U}$ of (4.21) is a sub pseudo-operad of Y. Defining relation (4.23) is obviously compatible with the pseudo-operad structure of $\widehat{U}$, thus the reduced collection $\{\mathsf{F}_k(n)\}_{n\geq 2}$ is a pseudo-operad as well. The operad structure on the full collection $\{\mathsf{F}_k(n)\}_{n\geq 1}$ is then given by formally adjoining the identity $1 \in \mathsf{F}_k(1) = *$ to the pseudo-operad $\{\mathsf{F}_k(n)\}_{n\geq 2}$, see **??**.

The map (4.19) induces, for the n-corolla $t(n)$, a Σ_n-equivariant morphism

$$\phi(n) : \mathring{\mathsf{F}}_k(n) \cong \mathring{\mathsf{F}}_k(t(n)) \xrightarrow{\iota_{t(n)}} \widehat{U}[t(n)] \hookrightarrow \widehat{U}(n) \xrightarrow{\text{proj.}} \mathsf{F}_k(n).$$

By the universal property of the free operad $\mathcal{F}(\mathring{\mathsf{F}}_k)$, the sequence $\{\phi(n)\}_{n\geq 2}$ gives rise to the desired continuous operadic map $\rho : \mathcal{F}(\mathring{\mathsf{F}}_k) \to \mathsf{F}_k$.

Let us prove that each $\rho(n)$ is a set isomorphism. This statement is clear for $n = 1$, since $\mathcal{F}(\mathring{\mathsf{F}}_k)(1) = \mathsf{F}_k(1) = *$, so assume $n \geq 2$. By definition,

$$\mathcal{F}(\mathring{\mathsf{F}}_k)(n) = \bigsqcup_{T \in \mathtt{Tree}_n^r} \mathring{\mathsf{F}}_k(T).$$

It is clear that, for $T \in \mathtt{Tree}_n^r$, the restriction $\rho|_{\mathring{\mathsf{F}}_k(T)}$ coincides with the composition

$$\mathring{\mathsf{F}}_k(T) \xrightarrow{\iota_T} \widehat{U}[T] \xrightarrow{\text{proj.}} \mathring{\mathsf{F}}_k(n),$$

where ι_T is the map (4.19). In other words, $\rho(n)$ sends an element of $\mathring{\mathsf{F}}_k(T)$ into the class of the corresponding primitive element. The fact that $\rho(n)$ is a set isomorphism then follows from the following statement whose proof is immediate.

Claim. *Suppose A is a set, $P \subset A$ a subset and let $f : A \to P$ be a map of sets such that $f|_P = \mathbb{1}_P$. Let $X := A/\sim$ with the relation $\sim$ given by*

$$A \ni a \sim f(a) \in P \subset A.$$

Then the composition $\rho : P \hookrightarrow A \xrightarrow{\text{proj.}} X$ is an isomorphism of sets.

In our concrete situation, we put $A = \widehat{U}(n)$, $P \subset \widehat{U}(n)$ to be the set of primitive elements and the map f induced by γ of (4.20). ■

Proposition 4.3.7 implies that, forgetting the topology, the compactification F_k is the free operad on the collection $\mathring{\mathsf{F}}_k$. The role of non-primitive elements in the quotient (4.22)

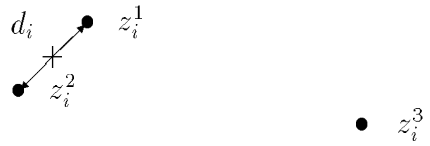


Figure 4.10: Normal configuration (z_i^1, z_i^2, z_i^3) for $i \gg 0$. The distance $d_i := |z_i^1 - z_i^2|$ is very small compared to $|z_i^1 - z_i^3|$ or $|z_i^2 - z_i^3|$. The $+$ denotes the 'center of gravity' of points z_i^1 and z_i^2.

is to introduce an appropriate topology or, in other words, to glue together the 'strata' $\{\mathring{\mathsf{F}}_k(T)\}_{T \in \mathtt{Tree}_n^r}$ forming the space $\mathsf{F}_k(n)$. This means that the set $\mathsf{F}_k(n)$ can be decomposed into the disjoint union of strata,

$$\mathsf{F}_k(n) = \bigcup_{T \in \mathtt{Tree}_n^r} \mathring{\mathsf{F}}_k(T). \tag{4.25}$$

The moduli space $\mathring{\mathsf{F}}_k(n)$ is a subset of $\mathsf{F}_k(n)$ corresponding, in the above decomposition, to the n-corolla $t(n)$. Equivalently, $\mathring{\mathsf{F}}_k(n)$ is the set of equivalence classes of points $\xi = (h, \zeta) \in \hat{U}(T)$ such that

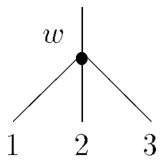
$$\mathfrak{z}(h) = \mathrm{rv}(T). \tag{4.26}$$

Example 4.3.8. Let us give some intuition behind formula (4.22) and explain why $\mathsf{F}_k(n)$ indeed compactifies the space $\mathring{\mathsf{F}}_k(n)$. Let us start with the case $k = 2$ and $n = 3$. Consider a sequence $\{x_i \in \mathring{\mathsf{F}}_2(3)\}_{i \geq 1}$ and show that it has a cluster point in $\mathsf{F}_2(3)$.

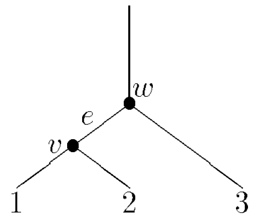
Choose the normal representative $(z_i^1, z_i^2, z_i^3) \in (\mathbb{R}^2)^3$ for each $x_i \in \mathring{\mathsf{F}}_2(3)$, $i \geq 1$. Since the subspace of all normal triples is compact, the sequence (z_i^1, z_i^2, z_i^3) has a cluster point (c^1, c^2, c^3) in $(\mathbb{R}^2)^3$, which is again a normal triple. If all c^1, c^2, c^3 are distinct we are done, because then the class $[c^1, c^2, c^3] \in \mathring{\mathsf{F}}_2(3)$ is clearly a cluster point of the sequence $\{x_i\}_{i \geq 1}$.

If the points c^1, c^2, c^3 are not distinct, then exactly two of them coincide, the case when all three points are the same is excluded by the second condition of (4.7). Suppose, without the loss of generality, that $c^1 = c^2$. This means that, after passing to a subsequence if necessary, the points z_i^1 and z_i^2 are very close for i big enough. The configuration looks as that shown on Figure 4.10.

The point (z_i^1, z_i^2, z_i^3) has several representatives in the 'big' space $\widehat{U}(3)$. One such a representative is the primitive one $\xi_i^p \in U[t(3)]$ attached to the 3-corolla $t(3)$:



with the vertex w colored by (z_i^1, z_i^2, z_i^3). We will consider another representative $\xi_i^{(\cdot\cdot)\cdot} \in U[T^{(\cdot\cdot)\cdot}]$ attached to the tree $T^{(\cdot\cdot)\cdot}$:



with the vertex w colored by the normalization of the configuration

$$(\frac{z_i^1 + z_i^2}{2}, z_i^3) \in (\mathbb{R}^2)^3$$

representing a point of $\mathring{\mathsf{F}}_2(\mathrm{inp}(w)) \cong \mathring{\mathsf{F}}_2(2)$, the vertex v colored by the normalization of the configuration

$$(z_i^1, z_i^2) \in (\mathbb{R}^2)^3$$

representing a point of $\mathring{\mathsf{F}}_2(\mathrm{inp}(v)) \cong \mathring{\mathsf{F}}_2(2)$ and the length function $h_i \in \mathrm{Met}(T^{(\cdot\cdot)\cdot})$ given by

$$h_i(e) := \frac{1}{3}\frac{|z_i^1 - z_i^2|}{|z_i^1 + z_i^2|}. \tag{4.27}$$

It then can be verified directly that

$$\omega_{T^{(\cdot\cdot)\cdot}}(\xi_i^{(\cdot\cdot)\cdot}) = (z_i^1, z_i^2, z_i^3) = \omega_{t(3)}(\xi_i^p)$$

for each i, therefore, by (4.23), the points $\xi_i^{(\cdot\cdot)\cdot}$ and ξ_i^p indeed represent the same element of the space $\mathsf{F}_2(3)$.

Intuitively, we introduced a new vertex v replacing two very close points points z_i^1 and z_i^2 by their 'center of gravity' $\frac{1}{2}(z_i^1 + z_i^2)$ and colored this new vertex by the 'microscopic' configuration (z_i^1, z_i^2). A rather complicated form (4.27) of $h_i(e)$ follows from our choice of the normalization. Observe that, for $|z_i^1 - z_i^2|$ very small,

$$h_i(e) \sim \frac{d_i}{6|z_i^1|}, \text{ with } d_i := |z_1^i - z_2^i|. \tag{4.28}$$

The representative $\xi_i^{(\cdot\cdot)\cdot}$ belongs to the set

$$U[T^{(\cdot\cdot)\cdot}] \subset Y[T^{(\cdot\cdot)\cdot}] \cong \mathring{\mathsf{F}}_2(2) \times \mathring{\mathsf{F}}_2(2) \times \mathbb{R}_{\geq 0} \cong S^1 \times S^1 \times \mathbb{R}_{\geq 0}.$$

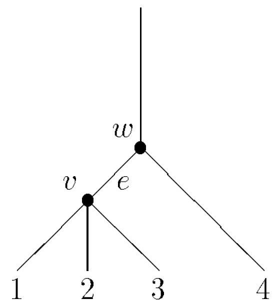
This set is not compact, but since the points z_i^1 and z_i^2 are, for $i \gg 0$, very close to each other, the third parameter in $\mathbb{R}_{\geq 0}$ representing the length $h_i(e)$ is, by (4.28), very small, so the sequence $\xi_i^{(\cdot\cdot)\cdot}$ has a cluster point $\bar{\xi}^{(\cdot\cdot)\cdot}$ in $\hat{U}[T^{(\cdot\cdot)\cdot}]$. The class $\bar{x}$ of $\bar{\xi}^{(\cdot\cdot)\cdot}$ in $\mathsf{F}_2(3)$ is then clearly a cluster point of the sequence $\{x_i\}_{i \geq 0}$.

Example 4.3.9. Let us make a similar discussion for the space $\mathring{\mathsf{F}}_2(4)$. So, let $\{x_i\}_{i\geq 1}$ be a sequence of points in $\mathring{\mathsf{F}}_2(4)$ and let $(z_i^1, z_i^2, z_i^3, z_i^4)$ be the normal representative of x_i, $i \geq 1$. Let us show again that the sequence $\{x_i\}_{i\geq 1}$ has a cluster point in $\mathsf{F}_2(4)$. Because the subspace of normal quadruples is compact, we may assume (after passing to a subsequence if necessary) that the sequence $\{(z_i^1, z_i^2, z_i^3, z_i^4)\}_{i\geq 1}$ converges to a point $(c^1, c^2, c^3, c^4) \in (\mathbb{R}^2)^4$. We distinguish four cases.

Case 1. All points c^1, c^2, c^3, c^4 are distinct. Then the class $[c^1, c^2, c^3, c^4] \in \mathring{\mathsf{F}}_2(4)$ is a cluster point of the sequence $\{x_i\}_{i\geq 1}$ and we are done.

Case 2. Exactly two of the points c^1, c^2, c^3, c^4 coincide, say $c^1 = c^2$. We proceed exactly as for $n = 3$, that is, we replace the points z_i^1 and z_i^2 by their 'center of gravity' and introduce a new vertex colored by the normalization of the 'microscopic' configuration (z_i^1, z_i^2).

Case 3. Exactly three of the points c^1, c^2, c^3, c^4, say $c^1 = c^2 = c^3$. This is the most difficult case. We represent the point x_i by some $\xi_i^{(\cdots)\cdot} \in U[T^{(\cdots)\cdot}]$ assigned to the tree $T^{(\cdots)\cdot}$:



with the vertex w colored by the normalization (p_i^1, p_i^2) of

$$\left(\frac{z_i^1 + z_i^2 + z_i^3}{3}, z^4\right),$$

the vertex v colored by the normalization (q_i^1, q_i^2, q_i^3) of (z_i^1, z_i^2, z_i^3), and the metric $h_i \in \mathrm{Met}(T^{(\cdots)\cdot})$ given by

$$h_i(e) := \frac{|2z_i^1 - z_i^2 - z_i^3|^2 + |2z_i^2 - z_i^1 - z_i^3|^2 + |2z_i^3 - z_i^1 - z_i^2|^2}{2|z_i^1 + z_i^2 + z_i^3|^2}.$$

The sequence $\{(p_i^1, p_i^2) \times (q_i^1, q_i^2, q_i^3) \times h_i(e)\}_{i\geq 1}$ of points of $\mathring{\mathsf{F}}_2(2) \times \mathring{\mathsf{F}}_2(3) \times \mathbb{R}_{\geq 0}$ has a cluster point

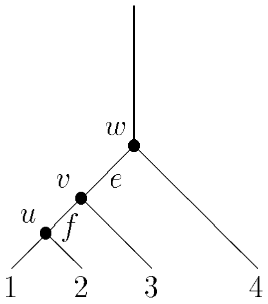
$$(\bar{p}^1, \bar{p}^2) \times (\bar{q}^1, \bar{q}^2, \bar{q}^3) \times 0 \in (\mathbb{R}^2)^2 \times (\mathbb{R}^2)^3 \times \mathbb{R}_{\geq 0}.$$

If all $\bar{q}^1, \bar{q}^2, \bar{q}^3$ are distinct, then the point $\bar{\xi}^{(\cdots)\cdot} = (0, \psi^{(\cdots)\cdot}) \in \hat{U}[T^{(\cdots)\cdot}]$, with

$$\psi^{(\cdots)\cdot}(w) := [\bar{p}^1, \bar{p}^2] \in \mathring{\mathsf{F}}_2(\mathrm{inp}(w)) \quad \text{and} \quad \psi^{(\cdots)\cdot}(v) := [\bar{q}^1, \bar{q}^2, \bar{q}^3] \in \mathring{\mathsf{F}}_2(\mathrm{inp}(v))$$

represents a cluster point of the sequence $\{x_i\}_{i\geq 1}$.

Suppose that not all $\bar{q}^1, \bar{q}^2, \bar{q}^3$ are distinct, say $q^1 = q^2$. We introduce yet another new vertex u replacing the tree $T^{(\cdots)\cdot}$ by $T^{((\cdot\cdot)\cdot)\cdot}$:



and define a representative $\xi^{((\cdot\cdot)\cdot)\cdot} \in U[T^{((\cdot\cdot)\cdot)\cdot}]$ by coloring the vertex w by the normalization of

$$\left(\frac{z_i^1 + z_i^2 + z_i^3}{3}, z^4\right),$$

the vertex v colored by the normalization of

$$\left(\frac{z_i^1 + z_i^2}{2}, z_i^3\right)$$

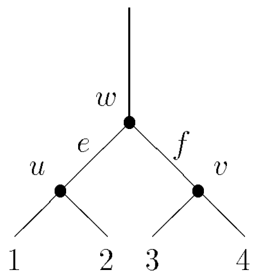
and the vertex v by the normalization of (z_i^1, z_i^2). We leave as an exercise to the reader to figure out how the length function $h_i \in \mathrm{Met}(T^{((\cdot\cdot)\cdot)\cdot})$ must be defined.

The sequence $\{\xi_i^{((\cdot\cdot)\cdot)\cdot}\}_{i\geq 1}$ has a cluster point $\bar{\xi}^{((\cdot\cdot)\cdot)\cdot}$ because

$$\begin{aligned} \xi_i^{((\cdot\cdot)\cdot)\cdot} \in Y[T^{((\cdot\cdot)\cdot)\cdot}] &\cong \mathring{\mathsf{F}}_2(\mathrm{inp}(w)) \times \mathring{\mathsf{F}}_2(\mathrm{inp}(v)) \times \mathring{\mathsf{F}}_2(\mathrm{inp}(u)) \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \\ &\cong (S^1)^3 \times (\mathbb{R}_{\geq 0})^2 \end{aligned}$$

where the last two 'noncompact' parameters, $h_i(e), h_i(f) \in \mathbb{R}_{\geq 0}$, can be assumed very small for $i \gg 0$. The last remaining case is:

Case 4. The last remaining case is $c^1 = c^2 \neq c^3 = c^4$ or a situation obtained from this one by a reindexation. We introduce the tree $T^{(\cdot\cdot)(\cdot\cdot)}$:



color the vertex w by the normalization of

$$\left(\frac{z_i^1 + z_i^2}{2}, \frac{z_i^3 + z_i^4}{2}\right)$$

the vertex u by the normalization of (z_i^1, z_i^2) and the vertex v by the normalization of (z_i^3, z_i^3). We again leave to the reader to figure out how the lengths $h_i(e)$ and $h_i(f)$ must be defined to get a representative $\xi_i^{(\cdot\cdot)(\cdot\cdot)}$ of x_i, $i \geq 1$, in

$$U[T^{(\cdot\cdot)(\cdot\cdot)}] \cong \mathring{\mathsf{F}}_2(\mathrm{inp}(w)) \times \mathring{\mathsf{F}}_2(\mathrm{inp}(v)) \times \mathring{\mathsf{F}}_2(\mathrm{inp}(u)) \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \cong (S^1)^3 \times (\mathbb{R}_{\geq 0})^2.$$

The first three relevant factors of the above space are compact, thus the sequence $\{\xi_i^{(\cdot\cdot)(\cdot\cdot)}\}_{i\geq 1}$ has a cluster point $\bar{\xi}^{(\cdot\cdot)(\cdot\cdot)} \in \hat{U}[T^{(\cdot\cdot)(\cdot\cdot)}]$ that represents a cluster point of the sequence $\{x_i\}_{i\geq 1}$.

The existence of a cluster point in $\mathsf{F}_k(n)$ of a sequence $\{x_i\}_{i\geq 1} \in \mathring{\mathsf{F}}_k(n)$ for arbitrary k and n can be proved using exactly the same arguments as in Example 4.3.8 and Example 4.3.9 by introducing a new vertex for each group of two or more points which are, for big i, close to each other and replacing these points by their center of gravity. Intuitively this can be imagined as applying a magnifying glass to 'microscopic' sub-configurations. Observe that we did not need the explicit formula for the metric h_i, we only needed to know that it is small for $i \gg 0$. We may thus formulate:

Lemma 4.3.10. *Let $k, n \geq 1$. Then each sequence $\{x_i\}_{i\geq 1} \in \mathring{\mathsf{F}}_k(n)$ has a cluster point in $\mathsf{F}_k(n)$.*

The main result of this section reads as follows.

Theorem 4.3.11. *Let $k, n \geq 1$. Then each $\mathsf{F}_k(n)$ is a compact smooth $(kn-k-1)$-dimensional manifold-with-corners and the topological operad F_k is in fact an operad in the category of manifolds-with-corners. The moduli space $\mathring{\mathsf{F}}_k(n)$ is an open dense subset of $\mathsf{F}_k(n)$.*

Proof. Let us show that each point $x \in \mathsf{F}_k(n)$ has a neighborhood U_x isomorphic to a neighborhood of a point in the cube I^{kn-k-1}, $I = [0,1]$. Recall that x has a unique primitive representative

$$(0,\psi) \in \widehat{U}[S],\ \psi \in \mathring{\mathsf{F}}_k(S),\ S \in \mathtt{Tree}^r_n,$$

with

$$\psi = \bigtimes_{w \in \mathrm{Vert}(S)} \psi_w, \quad \text{where } \psi_w \in \mathring{\mathsf{F}}_k(\mathrm{inp}(w)).$$

Let us choose, for each $w \in \mathrm{Vert}(S)$, neighborhoods U_w of ψ_w in $\mathring{\mathsf{F}}_k(\mathrm{inp}(w))$. Choose alsoand a positive number $\epsilon \in \mathbb{R}_{>0}$ and define the 'canonical neighborhood'

$$U_x := \left\{ \xi = (h,\zeta) \in Y[S];\ h(e) < \epsilon \text{ for } e \in \mathrm{Vert}(S) \text{ and } \zeta \in \bigtimes_{w \in \mathrm{Vert}(S)} U_w \subset \mathring{\mathsf{F}}_k(S) \right\}. \tag{4.29}$$

By a standard continuity argument, the neighborhoods U_w and the number ϵ can be choosen small enough so that U_x is a subset of $\widehat{U}[S]$. Define the map $u_x : U_x \to \mathsf{F}_k(n)$ as the composition

$$U_x \hookrightarrow \widehat{U}[S] \xrightarrow{\text{proj.}} \mathsf{F}_k(n).$$

Suppose that $\xi_i = (h_i, \zeta_i) \in U_x$, $i = 1,2$, be two distinct points. It is clear that if $u_x(\xi_1) = u_x(\xi_2)$, then $\mathfrak{Z}(h_1) = \mathfrak{Z}(h_2)$, so we may apply Lemma 4.3.5 to conclude that $\xi_1 = \xi_2$. This means that the map u_x is a monomorphism.

The map u_x is clearly continuous. It is also, by Lemma 4.3.6, an open map. So u_x is an isomorphism of U_x to its image $u_x(U_x) \subset \mathsf{F}_k(n)$.

To simplify the notation, let $N_w := k \cdot |\mathrm{inp}(w)| - k - 1$, for $w \in \mathrm{Vert}(S)$. Since $\mathring{\mathsf{F}}_k(\mathrm{inp}(w))$ is an N_w-dimensional smooth manifold, we may assume that the subset U_w of $\mathring{\mathsf{F}}_k(\mathrm{inp}(w))$ is diffeomorphic to an open subset of $\mathbb{R}^{N_w}$. Then U_x will clearly be diffeomorphic to a neighborhood of the point

$$(0, \ldots, 0, \tfrac{1}{2}, \ldots, \tfrac{1}{2}) \in I^{kn-k-1},$$

where the number of the zeroes is equal to the cardinality of the set $\mathrm{edg}(S)$.

Choosing the above data for each $x \in \mathsf{F}_k(n)$ we obtain an atlas $\mathfrak{U} = \{U_x, u_x\}_{x \in \mathsf{F}_k(n)}$. To prove that it defines a structure of a smooth manifold-with-corners one needs to investigate transition functions and prove that they are smooth. An explicit description of these functions is very clumsy, but the point is that they are combinations of maps ω_T (4.15), projections $\mathrm{proj} : \mathsf{F}(\mathbb{R}^k, m) \to \mathring{\mathsf{F}}_k(m)$ and normalizations $\mathrm{nor} : \mathring{\mathsf{F}}_k(m) \to \mathsf{F}(\mathbb{R}^k, m)$, for a choice of trees T and natural numbers m. From this it follows that they are smooth as desired.

It can be verified that each two distinct points x and y of $\mathsf{F}_k(n)$ admit mutually disjoint canonical neighborhoods (4.29) U_x and U_y which proves that the manifold $\mathsf{F}_k(n)$ is Hausdorff.

Let us prove that $\mathsf{F}_k(n)$ is compact. Since the topological space $\mathsf{F}_k(n)$ has obviously a countable basis, it is enough to prove that each sequence $\{x_i\}_{i \geq 1}$ of points of $\mathsf{F}_k(n)$ has a cluster point. We already know, by Lemma 4.3.10, that this is true when all $x_i \in \mathring{\mathsf{F}}_k(n)$. Let us consider the general case.

Let $(0, \psi_i) \in \hat{U}[S_i]$, $\psi_i \in \mathring{\mathsf{F}}_k(S_i)$, be a primitive representative of x_i, $S_i \in \mathtt{Tree}^r_n$. Since the set $\mathtt{Tree}^r_n$ is finite, there exists $S \in \mathtt{Tree}^r_n$ such that $S_i = S$ for infinitely many $i \geq 0$. So we may assume, passing to a subsequence if necessary, that $S_i = S$ for all i. Recall that $\psi_i(w) \in \mathring{\mathsf{F}}_k(\mathrm{inp}(w))$ denotes the ψ_i-color of $w \in \mathrm{Vert}(S)$.

By Lemma 4.3.10, the sequence $\{\psi_i(w)\}_{i \geq 1}$ has a cluster point in $\mathsf{F}_k(\mathrm{inp}(w))$, for each $w \in \mathrm{Vert}(S)$. This clearly implies that the sequence $\{\psi_i\}_{i \geq 1}$ has a cluster point

$$\bar{\psi} := \bigtimes_{w \in \mathrm{Vert}(S)} \bar{\psi}_w$$

in $\mathsf{F}_k(S) \supset \mathring{\mathsf{F}}_k(S)$. The operad structure γ_{F} of F_k induces, as usual, the contraction

$$\gamma_S : \mathsf{F}_k(S) \to \mathsf{F}_k(n)$$

along the tree S. Then the point $\bar{x} := \gamma_S(\bar{\psi})$ is a cluster point of the sequence $\{x_i\}_{i \geq 1}$.

To prove that $\mathring{\mathsf{F}}_k(n)$ is a dense subset of $\mathsf{F}_k(n)$ we observe that each canonical neighborhood U_x of a point $x \in \mathsf{F}_k(n)$ has a nonempty intersection with $\mathring{\mathsf{F}}_k(n)$. Indeed, U_x clearly contains a some $\xi = (h, \zeta)$ such that $h(e) > 0$ for all $e \in \mathrm{edg}(S)$. Then $\mathfrak{Z}(h) = \{\mathrm{rv}(S)\}$, thus, by (4.12), $[\xi] \in \mathring{\mathsf{F}}_k(n)$. ■

Example 4.3.12. Let us consider configurations of points on the real line. The space $\mathsf{F}(\mathbb{R}, n)$ is clearly, for each $n \geq 1$, a free Σ_n-space,

$$\mathsf{F}(\mathbb{R}, n) \cong \underline{\mathsf{Con}}(\mathbb{R}, n) \times \Sigma_n, \tag{4.30}$$

where

$$\underline{\mathsf{Con}}(\mathbb{R}, n) := \{(z^1, \ldots, z^n) \in \mathbb{R}^n;\ z^1 < \cdots < z^n\}.$$

Presentation (4.30) induces the factorization

$$\mathring{\mathsf{F}}_1(n) \cong \underline{\mathring{\mathsf{F}}}_1(n) \times \Sigma_n,$$

with

$$\underline{\mathring{\mathsf{F}}}_1(n) := \underline{\mathsf{Con}}(\mathbb{R}, n)/\mathrm{Aff}(\mathbb{R}) \cong \{(z^1, \ldots, z^n) \in \mathbb{R}^n;\ 0 = z^1 < \cdots < z^n = 1\},$$

which is the open $(n-2)$ simplex $\mathring{\Delta}^{n-2}$. There is a corresponding decomposition of the compactification:

$$\mathsf{F}_1(n) \cong \underline{\mathsf{F}}_1(n) \times \Sigma_n.$$

The space $\underline{\mathsf{F}}_1(n)$ is, as a manifold-with-corners, isomorphic to the associahedron K_n. In fact, the associahedron can be obtained by blowing-up some faces of the simplex Δ^{n-2} or, equivalently, by a truncation of $\mathring{\Delta}^{n-2}$, see [118, Appendix B].

Recall **??** that each K_n is a cellular chain complex, with cells indexed by reduced planar n-trees. This decomposition corresponds to the stratification (4.25) of the compactification.

The collection $\underline{\mathsf{F}}_1 = \{\underline{\mathsf{F}}_1(n)\}_{n\geq 1}$ is a non-Σ operad, isomorphic to the non-Σ operad of the associahedra $K = \{K_n\}_{n\geq 1}$.

Example 4.3.13. The operad $\mathsf{F}_2 = \{\mathsf{F}_1(n)\}_{n\geq 1}$ was intensively studied as a candidate for a solution of the Deligne conjecture, see also **??**. Getzler and Jones [36] constructed a cell decomposition of $\mathring{\mathsf{F}}_2$ compatible with the operad structure, induced by the Fox-Neuwirth decomposition of the collection of open parts $\mathring{\mathsf{F}}_2 = \{\mathring{\mathsf{F}}_2(n)\}_{n\geq 1}$. There was, unfortunately, an error in their construction but, as shown by Voronov in [132], there still exists a filtration compatible with the operad structure and extending the Fox-Neuwirth decomposition of the open parts.

configuration $a = (a^1, a^2)$: configuration $b_1 = (b_1^1, b_1^2, b_1^3)$: configuration $b_2 = (b_2^1, b_2^2)$:

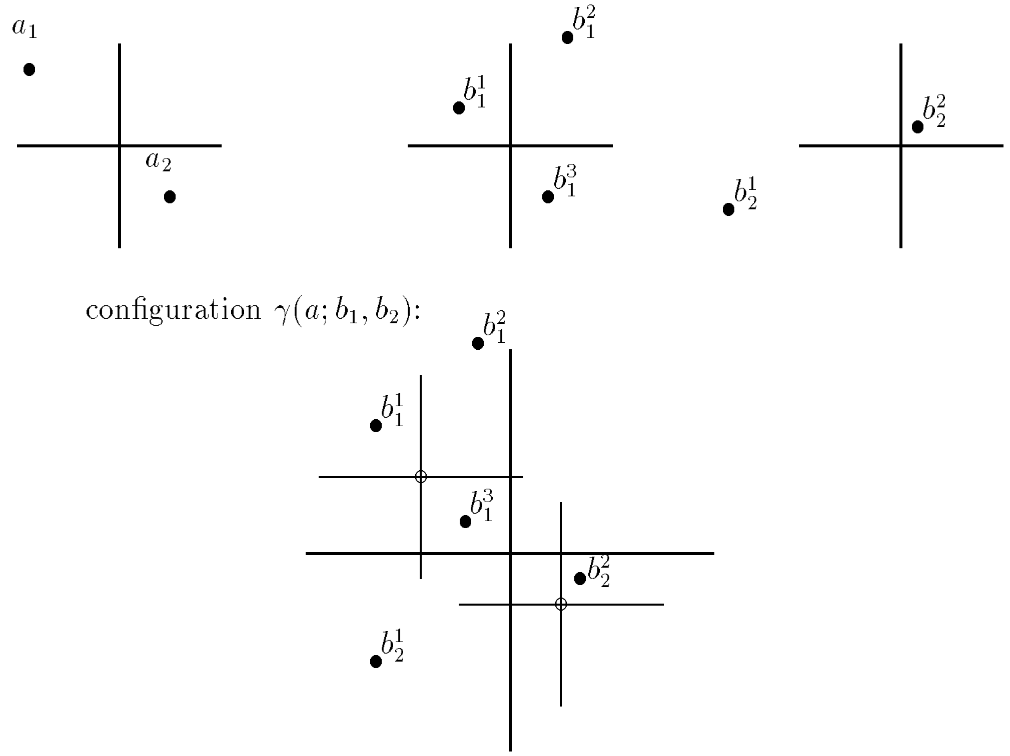


Figure 4.11: The construction of $\gamma(a; b_1, b_2) \in \mathsf{F}(\mathbb{R}^2, 5)$ from $a \in \mathsf{F}(\mathbb{R}^2, 2)$, $b_1 \in \mathsf{F}(\mathbb{R}^2, 3)$ and $b_2 \in \mathsf{F}(\mathbb{R}^2, 2)$.

4.3.1. Appendix.

In this appendix we describe a partial operad structure on the collection

$$\mathsf{F}(\mathbb{R}^k) = \{\mathsf{F}(\mathbb{R}^k, n)\}_{n \geq 1}.$$

We need to specify, for each $a = (a^1, \ldots, a^l) \in \mathsf{F}(\mathbb{R}^k, l)$ and $b_i \in \mathsf{F}(\mathbb{R}^k, m_i)$, the value of the 'composition map' $\gamma(a; b_1, \ldots, b_l) \in \mathsf{F}(\mathbb{R}^k, m_1 + \cdots + m_l)$. This can be done by putting

$$\gamma(a; b_1, \ldots, b_l) := ((\underbrace{a^1, \ldots, a^1}_{m_1 \text{ times}}) + b_1, (\underbrace{a^2, \ldots, a^2}_{m_2 \text{ times}}) + b_2, \ldots, (\underbrace{a^l, \ldots, a^l}_{m_l \text{ times}}) + b_l).$$

The configuration $\gamma(a; b_1, \ldots, b_l)$ may be viewed as the superposition of the configurations $T_{a_1}(b_1), \ldots, T_{a_l}(b_l)$, where $T_a(-)$ means, just here and now, the translation by a vector $a \in \mathbb{R}^k$. This process is visualized on Figure 4.11.

We encourage the reader to verify that all the axioms of an operad are satisfied. The only drawback is that $\gamma(a; b_1, \ldots, b_l)$ need not necessarily be an element of the configuration space $\mathsf{F}(\mathbb{R}^k, m_1 + \cdots + m_l)$, because the components of $\gamma(a; b_1, \ldots, b_l)$ need not be different. Thus the structure map is defined only for *some* elements of

$$\mathsf{F}(\mathbb{R}^k, l) \times \mathsf{F}(\mathbb{R}^k, m_1) \times \cdots \times \mathsf{F}(\mathbb{R}^k, m_l).$$

Such an object may be called a *partial operad*.

4.5. Compactification of configurations of points in a manifold.

Version of September 14

The aim of this section is to modify the approach of Section 4.3 and review a construction of a compactification $\overline{\mathsf{Con}}(M, n)$ of the configuration space $\mathsf{F}(M, n)$ of n distinct labelled points in a complete k-dimensional Riemannian manifold M, originally due to S. Axelrod and I.M. Singer [1]. As explained by Proposition 4.5.5 and the remark following this proposition, the collection $\overline{\mathsf{Con}}(M) = \{\overline{\mathsf{Con}}(M, n)\}_{n\geq 1}$ has, for a general M, no 'reasonable' algebraic structure, because there is no canonical identification of the tangent space T_xM at a point $x \in M$ with the 'model space' $\mathbb{R}^k$. Such an identification exists if M is parallelizable, the collection $\overline{\mathsf{Con}}(M)$ is then a right module over the operad $\mathsf{F}_k = \{\mathsf{F}_k(n)\}_{n\geq 1}$ constructed in **??** (Theorem 4.5.7).

More suitable for a non-parallelizable manifold M is the framed configuration space $f\mathsf{F}(M, n)$ introduced in Definition 4.1.9. We construct a compactification $\mathfrak{f}\overline{\mathsf{Con}}(M, n)$ of this space and show that the collection $\mathfrak{f}\overline{\mathsf{Con}}(M) = \{\mathfrak{f}\overline{\mathsf{Con}}(M, n)\}_{n\geq 1}$ forms a right module over a framed version $\mathfrak{f}\mathsf{F}_k$ of the operad F_k (Theorem 4.5.10).

This section is very technical and consists mostly of constructions of the above mentioned objects.

Compactification of $\mathsf{F}(M, n)$. Let us describe a compactification of the (unframed) configuration space $\mathsf{F}(M, n)$ of n distinct labelled points in a complete Riemannian manifold M. The steps of our construction will be analogous to the steps of the construction of the compactification $\mathsf{F}_V(n)$ in Section 4.3.

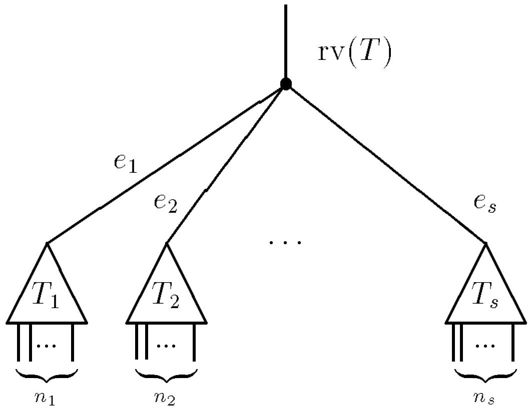
Instead of the set $\mathtt{Tree}^r_n$ of reduced rooted n-trees, we use here the set $\mathtt{Tree}^e_n$ of all n-trees such that all vertices except possibly the vertex adjacent to the root are at least binary (the root vertex may have arity 1). In other words, $\mathtt{Tree}^e$ denotes the set of trees with a *special vertex* – the root – such that all vertices which are not special are of arity ≥ 2. Observe that while $\mathtt{Tree}^r_1 = \emptyset$, $\mathtt{Tree}^e_1 = \{\phi\}$, the 1-corolla with one vertex of arity 1. It will be useful to think of a tree $T \in \mathtt{Tree}^e_n$ as obtained by grafting subtrees $T_i \in \mathtt{Tree}^r_{n_i}$ to the s-corolla, for $s \leq n$ and some (not necessarily all) $1 \leq i \leq s$, as indicated in Figure 4.12.

Let $\mathrm{Met}(T)$ denote, as in Section 4.3, the space of all metrics $h : \mathrm{edg}(T) \to \mathbb{R}_{\geq 0}$ on the set of (inner) edges of a tree T. We will also need the following notation.

Notation 4.5.1. *For a tree $T \in \mathtt{Tree}^e_n$, let $\mathrm{ir}(T)$ (inputs of root) denote the set $\mathrm{inp}(\mathrm{rv}(T))$ of input edges of the root vertex $\mathrm{rv}(T)$ of T. We also denote by $\mathrm{iVert}(T)$ the set*

$$\mathrm{iVert}(T) := \mathrm{Vert}(T) - \{\mathrm{rv}(T)\}$$

(the set of inner vertices).



A tree $T \in \mathtt{Tree}^e_n$ decomposed to the s-corolla and trees $T_i \in \mathtt{Tree}^e_{n_i}$, $1 \leq i \leq s$. The set $\mathrm{ir}(T)$ equals $\{e_1, \ldots, e_s\}$. Some of the trees T_i may be absent, as is T_2 in the following tree with $s = 3$:

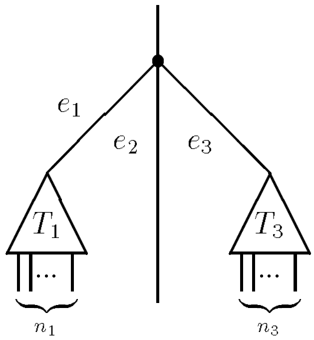


Figure 4.12: Examples of trees from $\mathtt{Tree}^e_n$.

For any vertex $v \in \mathrm{iVert}(T)$, there exists a unique path in T connecting v and $\mathrm{rv}(T)$; this path contains exactly one element of $\mathrm{ir}(T)$ which we denote by $\mathrm{pa}(v)$ (path of v). Similarly, for each $1 \leq i \leq n$ there exists a unique edge $\mathrm{pa}(i) \in \mathrm{ir}(T)$ (path of i) such that $\mathrm{pa}(i)$ lies on the path from the ith input leaf of T to the root. Sometimes we will write more explicitly $\mathrm{pa}_T(v)$ and $\mathrm{pa}_T(i)$ instead of $\mathrm{pa}(v)$ and $\mathrm{pa}(i)$.

Let us come back to the manifold M. Since we assume M to be Riemannian, the tangent space $T_x(M)$ at each point $x \in M$ has an $\mathbb{R}$-linear positive definite metric, so we may consider the moduli spaces $\mathring{\mathsf{F}}_{T_x(M)}(n)$ and their compactifications $\mathsf{F}_{T_x(M)}(n)$, $n \geq 1$, see **??** for the definition. We simplify the clumsy notation $\mathsf{F}_{T_x(M)}(n)$ by introducing

$$\mathsf{F}_x(n) := \mathsf{F}_{T_x(M)}(n), \text{ for } x \in M \text{ and } n \geq 1, \tag{4.31}$$

and similarly, for the open parts, $\mathring{\mathsf{F}}_x(n) := \mathring{\mathsf{F}}_{T_x(M)}(n)$. The system $\{\mathring{\mathsf{F}}_x(n)\}_{x \in M}$ can be clearly pasted to form a smooth fibration

$$\mathsf{f}_n : \mathring{\mathsf{F}}(n) \to M$$

with fiber $\mathsf{f}_n^{-1}(x) = \mathring{\mathsf{F}}_x(n)$, $x \in M$. The standard extension trick enables one to consider similar fibrations $\mathsf{f}_S : \mathring{\mathsf{F}}(S) \to M$ with an arbitrary nonempty finite set S.

For each $T \in \mathtt{Tree}_n^e$, there is a map $a_T : \mathsf{F}(M, \mathrm{ir}\,(T)) \to M^{\mathrm{iVert}(T)}$ given by

$$a_T(\vec{x})(v) := x^{\mathrm{pa}(v)}$$

for $v \in \mathrm{iVert}(T)$ and $\vec{x} = \{x^e\}_{e \in \mathrm{ir}\,(T)} \in \mathsf{F}(M, \mathrm{ir}\,(T))$. Using the schematic picture Figure 4.12, $a_T(\vec{x})$ can be seen as the function whose value at $v \in \mathrm{Vert}(T_i)$ is x^{e_i}, $1 \leq i \leq s$. Let $\mathsf{F}(M)(T)$ denote the pullback of the diagram

$$\begin{array}{ccc} & & \displaystyle\bigtimes_{v \in \mathrm{iVert}(T)} \mathring{\mathsf{F}}(\mathrm{inp}(v)) \\ & & \Big\downarrow \displaystyle\bigtimes_{v \in \mathrm{iVert}(T)} \mathsf{f}_{\mathrm{inp}(v)} \\ \mathsf{F}(M, \mathrm{ir}\,(T)) & \xrightarrow{\ a_T\ } & M^{\mathrm{iVert}(T)}. \end{array}$$

The set $\mathsf{F}(M)(T)$ is the set of all colorings of the tree T such that the root is colored by an element $\vec{x} = \{x^e\}_{e \in \mathrm{ir}\,(T)}$ of the configuration space $\mathsf{F}(M, \mathrm{ir}\,(T))$ while the remaining vertices $v \in \mathrm{iVert}(T)$ (if any) are colored by elements of the space $\mathring{\mathsf{F}}_{x^{\mathrm{pa}(v)}}(\mathrm{inp}(v))$. Let us define, for $T \in \mathtt{Tree}_n^e$,

$$Z[T] := \mathrm{Met}(T) \times \mathsf{F}(M)(T),$$

compare (4.13). Elements of $Z[T]$ are sequences

$$\eta = \{\lambda_v;\ v \in \mathrm{Vert}(T)\} \tag{4.32}$$

with

$$\lambda_{\mathrm{rv}(T)} = \vec{x} = \{x^e\}_{e \in \mathrm{ir}\,(T)} \in \mathsf{F}(M, \mathrm{ir}\,(T))$$

and, for $v \in \mathrm{iVert}(T)$,

$$\lambda_v = (t_v, [\vec{z}_v]) \in \mathbb{R}_{\geq 0} \times \mathring{\mathsf{F}}_{x^{\mathrm{pa}(v)}}(\mathrm{inp}(v)),$$

compare (4.14).

For each $1 \leq i \leq n$, there is in the tree T a unique path from the i-th input leaf to the root, as in Figure 4.8 of Section 4.3. We then define a map $\varphi_i : Z[T] \to M$ by the formula

$$\varphi_i(\eta) := \exp_{x^{e_s}}(t_{v_{s-1}} z_{v_{s-1}}^{e_{s-1}} + t_{v_{s-2}} t_{v_{s-1}} z_{v_{s-2}}^{e_{s-2}} + \cdots + t_{v_2} \cdots t_{v_{s-1}} z_{v_2}^{e_2} + t_{v_1} \cdots t_{v_{s-1}} z_{v_1}^{i}), \tag{4.33}$$

where $\exp_{x^{e_s}} : T_{x^{e_s}}(M) \to M$ is the exponential map at x^{e_s} whose existence is guaranteed by the completeness of M.

The idea behind this formula is similar to that of (**??**), that is, $\varphi_i(\eta)$ is a superposition of $x^{e_s} \in M$ with the image under the exponential map of $z^{e_{s-1}}_{v_{s-1}}$ scaled by $t_{v_{s-1}}$, $z^{e_{s-2}}_{v_{s-2}}$ scaled by $t_{v_{s-2}} t_{v_{s-1}}$, ..., $z^i_{v_{s-1}}$ scaled by $t_{v_1} \cdots t_{v_{s-1}}$. Finally, let $\varphi_T : Z[T] \to M^n$ be given as

$$\varphi_T(\eta) := (\varphi_1(\eta), \dots, \varphi_n(\eta)) \in M^n,$$

to be compared with the definition (4.15) of the map $\omega_T : Y[T] \to (\mathbb{R}^k)^n$. The subset $W[T] \subset Z[T]$ of *composable elements* (an analog of the set $U[T]$ of 4.16) is defined by

$$W[T] := \{\eta \in Z[T]; \text{ all } \varphi_i(\eta) \in M \text{ are distinct, } 1 \le i \le n\}. \tag{4.34}$$

Each $\eta = (h, \phi) \in Z[T]$ with a metric $h \in \mathrm{Met}(T)$ determines the 'zero set' $\mathfrak{Z}(h) \subset \mathrm{Vert}(T)$ as in (4.12). Recall that the tree T then decomposes into disjoint subtrees $\{T_w\}_{w \in \mathfrak{Z}(h)}$, see also Figure **??**. For $w \in \mathfrak{Z}(h)$ and η as above we have, as in (4.17), obvious restrictions

$$r_w(\eta) \in \begin{cases} Z[T_w], & \text{for } w = \mathrm{rv}(T), \text{ and} \\ Y_{x^{\mathrm{pa}_T(w)}}[T_w], & \text{for } w \in \mathfrak{Z}(h),\ w \neq \mathrm{rv}(T). \end{cases} \tag{4.35}$$

where

$$Y_{x^{\mathrm{pa}_T(w)}}[T_w] := Y_{T_{x^{\mathrm{pa}_T(w)}}(M)}[T_w]$$

with $Y_V[T]$, for a vector space V, introduced in **??**. The meaning of (4.35) is the following. The decomposition $\{T_w\}_{w \in \mathfrak{Z}(h)}$ is the decomposition into biggest subtrees such that the metric h is possitive on all (inner) edges of T_w, for all $w \in \mathfrak{Z}(h)$. The tree T_w with $w = \mathrm{rv}(T)$ must be considered separately, since it contains the special vertex, $T_w \in \mathtt{Tree}^e$. The restriction of the 'coloring' η is then an element of $Z[T_w]$. The tree T_w with $w \neq \mathrm{rv}(T)$ has no special vertex, so all its vertices v are 'colored' by $\mathring{\mathsf{F}}_{x^{\partial(v)}}(\mathrm{inp}(v))$, thus the restriction of η to T_w is an element of $Y_V[T]$ with $V = T_{x^{\partial(w)}}(M)$. Thus the definition of $r_w(\eta)$ makes sense for all $w \in \mathfrak{Z}(h)$.

The *extended set* $\widehat{W}[T]$ (an analog of the extended set $\hat{U}[T]$ of (4.18)) is the set of all $\eta \in Z[T]$ such that all restrictions $r_w(\eta)$ are composable, that is

$$r_w(\eta) \in \begin{cases} W[T_w], & \text{for } w = \mathrm{rv}(T), \text{ and} \\ U_{x^{\mathrm{pa}_T(w)}}[T_w], & \text{for } w \in \mathfrak{Z}(h),\ w \neq \mathrm{rv}(T), \end{cases} \tag{4.36}$$

where, in accord with our conventions, we denoted

$$U_{x^{\mathrm{pa}_T(w)}}[T_w] := U_{T_{x^{\mathrm{pa}_T(w)}}(M)}[T_w]$$

with the set $U_V[T_w]$ defined, for a vector space V, in (**??**).

As in Section 4.3, there are special *primitive* elements of $Z[T] = \mathrm{Met}(T) \times \mathsf{F}(M)(T)$ of the form $(0, \phi)$, where 0 denotes the trivial metric and $\phi \in \mathsf{F}(M)(T)$. It is clear that each such element belongs to $\widehat{W}[T]$.

Let T_h denote, as in Section 4.3, the tree obtained from T by shrinking each subtree T_w, $w \in \mathfrak{Z}(T)$, to a vertex which we denote again by w. We would like, as in (4.20), to assign to each $\eta \in \widehat{W}[T]$ some $\delta(\eta) \in \mathsf{F}(M)(T_h)$. The formula

$$\delta(\eta) := \vec{y} \times \bigtimes_{w \in \mathrm{iVert}(T_h)} [\omega_{T_w}(r_w(\eta))] \tag{4.37}$$

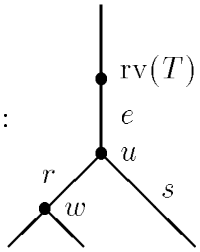
with

$$\vec{y} := \varphi_{T_{\mathrm{rv}(T)}}(r_{\mathrm{rv}(T)}(\eta)) \in \mathsf{F}(M, \mathrm{ir}(T_h)) \tag{4.38}$$

analogous to (**??**) unfortunately does not define an element of $\mathsf{F}(M)(T)$. The point is that $[\omega_{T_w}(r_w(\eta))] \in \mathring{\mathsf{F}}_{x^{\mathrm{pa}_T(w)}}(\mathrm{inp}_{T_h}(w))$ while it must be an element of $\mathring{\mathsf{F}}_{y^{\mathrm{pa}_{T_h}(w)}}(\mathrm{inp}_{T_h}(w))$. The following example shows that the points $x^{\mathrm{pa}_T(w)}$ and $y^{\mathrm{pa}_{T_h}(w)}$ are, in general, different.

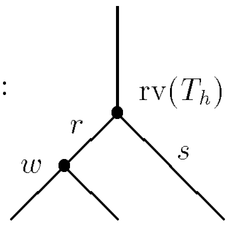
Example 4.5.2. Let us consider the following tree $T \in \mathtt{Tree}_3^e$:

T :



with $\mathrm{ir}(T) = \{e\}$, $\mathrm{inp}(u) = \{r, s\}$ and $\mathrm{pa}_T(w) = e$. Define a metric $h \in \mathrm{Met}(T)$ by $h(e) := t > 0$ and h trivial on remaining edges. Then $\mathfrak{Z}(h) = \{\mathrm{rv}(T), r\}$, $T_{\mathrm{rv}(T)}$ is the subtree with vertices $\mathrm{rv}(T)$ and u, and T_r is the subtree (2-corolla) with vertex r. The corresponding tree $T_h \in \mathtt{Tree}_3^e$ is obtained by collapsing $T_{\mathrm{rv}(T)}$ and T_r to corollas,

T_h :



where we used the letters s and r to denote the edges corresponding to the edges of T with the same name. Clearly $\mathrm{ir}(T_h) = \{r, s\}$ and $\mathrm{pa}_{T_h}(w) = r$.

Let us consider $\eta \in Z[T]$ (notation of (4.32)) with $\lambda_u = (t, [z^r, z^s])$, where (z^r, z^s) is a normal pair of points of $T_x M$, $x := x^e$. For this η we clearly have $\mathrm{pa}_T(w) = e$, therefore $x^{\mathrm{pa}_T(w)} = x$.

On the other hand, the element $\vec{y} = (y^r, y^s)$ of (4.38) is given by

$$y^r = \exp_x(tz^r) \text{ and } y^s = \exp_x(tz^s),$$

as follows from (4.33). We conclude that, at least for some if not for all $t > 0$,

$$x^{\mathrm{pa}_T(w)} = x \neq \exp_x(tz^r) = y^{\mathrm{pa}_{T_h}(w)}. \tag{4.39}$$

So we need to move the factor $[\omega_{T_w}(r_w(\eta))]$ in (4.37) from the point $x^{\mathrm{pa}_T(w)}$ to $y^{\mathrm{pa}_{T_h}(w)}$. We use the 'parallel transport' $\Phi_{x,y} : T_x(M) \to T_y(M)$ defined, for $x \in M$, $a \in T_x(M)$ and $y := \exp_x(a)$ by

$$\Phi_{x,y}(\xi) := \left.\frac{d}{ds}\right|_{s=0} \exp_x(a + s\xi),\ \xi \in T_x(M),$$

compare [1, (5.80)]. Strictly speaking, the notation $\Phi_{x,y}$ is not correct because there might be several a's with $y := \exp_x(a)$, but in our applications the a will always be as explicit as in (4.39) where $a = tz^s$.

Parallel transport can clearly be used to move all objects functorially depending on the tangent spaces of M. In particular, parallel transport induces the map

$$\Phi_{x,y} : \mathring{\mathsf{F}}_x(S) \to \mathring{\mathsf{F}}_y(S),$$

for an arbitrary finite set S. The correct formula for $\delta : \widehat{W}[T] \to \mathsf{F}(M)(T_h)$ is

$$\delta(\eta) := \vec{y} \times \bigtimes_{w \in \mathrm{iVert}(T_h)} \Phi_{x^{\mathrm{pa}_T(w)},\, y^{\mathrm{pa}_{T_h}(w)}}[\omega_{T_w}(r_w(\eta))]. \tag{4.40}$$

We then proceed exactly as in Section 4.3, that is, we define

$$\overline{\mathsf{Con}}(M,n) := \widehat{W}(n)/\sim,$$

where

$$\widehat{W}(n) := \bigsqcup_{T \in \mathtt{Tree}_n^e} \widehat{W}[T]$$

and the relation $\sim$ given by

$$\widehat{W}(n) \supset \widehat{W}[T] \ni \eta \sim (0, \delta(\eta)) \in \widehat{W}[T_h] \subset \widehat{W}(n),$$

for $T \in \mathtt{Tree}_n^e$ and $n \geq 1$. The following theorem can be proved exactly as Theorem 4.3.11 of Section **??**.

Proposition 4.5.3. *The space $\overline{\mathsf{Con}}(M,n)$ is a smooth manifold-with-corners of dimension kn containing the configuration space $\mathsf{F}(M,n)$ as an open dense subset. It is compact if and only if M is.*

As proved in [83], the space $\overline{\mathsf{Con}}(M,n)$ coincides with the Axelrod-Singer compactification of the configuration space [1]. As in (4.25), the set $\overline{\mathsf{Con}}(M,n)$ is the disjoint union of strata,

$$\overline{\mathsf{Con}}(M,n) = \bigcup_{T \in \mathtt{Tree}_n^e} \mathsf{F}(M)(T),\ n \geq 1. \tag{4.41}$$

Example 4.5.4. It is immediate to see that $\overline{\mathsf{Con}}(M,1) \cong M$. In fact, decomposition (4.41) consist, for $n = 1$, of a single strata corresponding to the 1-corolla ϕ,

$$\mathsf{F}(M)(\phi) \cong M.$$

So even $\overline{\mathsf{Con}}(M,1)$ is not compact if M is not. Loosely speaking, the 'compactification' $\overline{\mathsf{Con}}(M,n)$ takes care of points that come close together, not of those that 'diverge to infinity' in M. Therefore a more appropriate name for $\overline{\mathsf{Con}}(M,n)$ would be a 'resolution of diagonals' (following [39]) or a 'partial compactification.'

Let us try to understand which kind of module structure the collection $\overline{\mathsf{Con}}(M) = \{\overline{\mathsf{Con}}(M,n)\}_{n\geq 1}$ may possibly have. Observe first that there is a *'blowing down' map* (terminology borrowed from algebraic geometry)

$$\mathrm{bld} : \overline{\mathsf{Con}}(M,n) \to M^{\times n},\ \mathrm{bld}(u) = (\mathrm{bld}_1(u), \ldots, \mathrm{bld}_n(u)),\ n \geq 1. \tag{4.42}$$

The most efficient way to define this map is to say that it is an extension of the inclusion $\mathsf{F}(M,n) \hookrightarrow M^{\times n}$ to the compactification $\overline{\mathsf{Con}}(M,n)$. This extension is necessarily unique, but its existence must be proved.

In terms of primitive representatives, the blowing-down map is given as follows. First, define $\mathrm{bld}_T : \mathsf{F}(M)(T) \to M^n$ by

$$\mathrm{bld}_T\left(\{x^e\}_{e\in \mathrm{ir}(T)} \times \bigtimes_{v\in \mathrm{iVert}(T)} [\vec{z}_v]\right) := (x^{\mathrm{pa}(1)}, \ldots, x^{\mathrm{pa}(n)}), \tag{4.43}$$

where $\mathrm{pa}(i)$, $1 \leq i \leq n$, was introduced in 4.5.1. Then we put $\mathrm{bld}(0,\phi) := \mathrm{bld}_T(\phi)$.

For each $x \in M$ we have the operad $\mathsf{F}_x := \mathsf{F}_{T_x(M)}$. This collection in fact forms a 'fibration'

$$\mathsf{f} : \mathsf{F} \to M,\ \mathsf{f} = \{\mathsf{f}_n : \mathsf{F}(n) \to M\}_{n\geq 1}$$

whose fiber over $x \in M$ is the operad F_x. For each $n \geq 1$ and $k_1, \ldots, k_n \geq 1$, let $X(n; k_1, \ldots, k_n)$ be the pullback of the diagram

$$\begin{tikzcd} & \bigtimes_{1\leq i\leq n} \mathsf{F}(k_i) \arrow[d, "\bigtimes_{1\leq i\leq n} \mathsf{f}_{k_i}"] \\ \overline{\mathsf{Con}}(M,n) \arrow[r, "\mathrm{bld}"] & M^{\times n} \end{tikzcd}$$

Elements of the space $X(n; k_1, \ldots, k_n)$ are arrays $(u; v_1, \ldots, v_n)$, where $u \in \overline{\mathsf{Con}}(M,n)$ and $v_i \in \mathsf{F}_{\mathrm{bld}_i(u)}(k_i)$, for $1 \leq i \leq n$. The algebraic structure of the collection $\overline{\mathsf{Con}}(M) = \{\overline{\mathsf{Con}}(M,n)\}_{n\geq 1}$ is described in the following proposition, which can be proved by arguments similar to those of Proposition **??**.

Proposition 4.5.5. *For each $n \geq 1$ and $k_1, \ldots, k_n \geq 1$, there exists a natural 'composition'*

$$\begin{aligned} \nu : X(n; k_1, \ldots, k_n) &\longrightarrow \overline{\mathsf{Con}}(M, k_1 + \cdots + k_n), \\ X(n; k_1, \ldots, k_n) \ni (u; v_1, \ldots, v_n) &\longmapsto \nu(u; v_1, \ldots, v_n) \in \overline{\mathsf{Con}}(M, k_1 + \cdots + k_n) \end{aligned}$$

satisfying obvious associativity, equivariance and unit axioms analogous to that of a right module over an operad.

Remark 4.5.6. A proper name for the structure on $\overline{\mathsf{Con}}(M)$ described in Proposition 4.5.5 would probably be a *colored* right module over the colored operad F, with a smooth set of colors M. While in an ordinary module N over an ordinary operad $\mathcal{P}$, each 'input' of $u \in N$ is acted on by elements of the same operad $\mathcal{P}$, in this case the 'i-th input' of $\overline{\mathsf{Con}}(M, n)$ is acted on by elements of the operad $\mathsf{F}_{\mathrm{bld}_i(u)}$ whose 'color' $\mathrm{bld}_i(u) \in M$ depends on both i and (smoothly) also on $u \in \overline{\mathsf{Con}}(M, n)$. Of course, all operads $\mathsf{F}_{\mathrm{bld}_i(u)}$ are isomorphic to F_k but, in general, noncanonically. One way to obtain a canonical identification is to assume that the manifold M is parallelizable, as in the following theorem.

Theorem 4.5.7. *Let M be a complete parallelizable Riemannian manifold. Then the collection $\overline{\mathsf{Con}}(M) = \{\overline{\mathsf{Con}}(M, n)\}_{n \geq 1}$ is a right F_k-module. The module structure is functorial up to a choice of the trivialization of the tangent bundle of M.*

Proof. The theorem will follow from a more general Theorem 4.5.10. ■

Example 4.5.8. Consider the configuration space of points on the unit cycle S^1. Clearly

$$\mathsf{F}(S^1, n) \cong \underline{\mathsf{Con}}(S^1, n) \times S^1 \times \Sigma_n/\mathbb{Z}_n, \tag{4.44}$$

where $\underline{\mathsf{Con}}(S^1, n)$ is the space of all n-tuples $(x^1, \ldots, x^n)$ of points of $S^1 = \{z \in \mathbb{C};\ |z| = 1\}$ such that $x^1 = 1$ and

$$0 < \arg(x^2) < \arg(x^3) < \cdots < \arg(x^n) < 2\pi,$$

see Figure 4.13. In (4.44), the group $\mathbb{Z}_n$ is embedded to Σ_n as the subgroup of cyclic permutations. Factorization (4.44) induces the decomposition of the compactification

$$\overline{\mathsf{Con}}(S^1, n) = W_n \times S^1 \times \Sigma_n/\mathbb{Z}_n,$$

where W_n, the compactification of $\underline{\mathsf{Con}}(S^1, n)$, is the compact $(n-1)$-dimensional polyhedron introduced by R. Bott and C. Taubes [11, page 5249] in connection with the study of nonperturbative link invariants (but denoted there by W_{n-1}), and recently dubbed, in [118], the *cyclohedron*. The polyhedron W_1 is just the point, W_2 is the interval, W_3 is the hexagon:

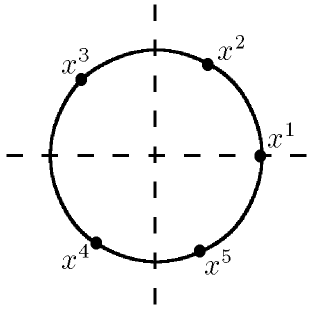


Figure 4.13: A configuration from $\underline{\mathsf{Con}}(S^1, 5)$.

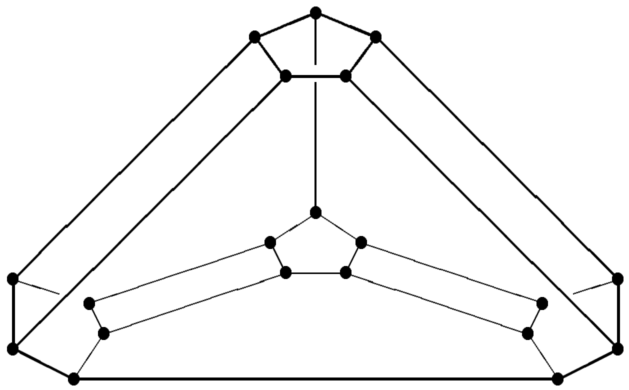
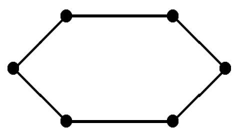


Figure 4.14: The cyclohedron W_4.



and the 3-dimensional polyhedron W_4 is portraited in Figure 4.14.

Since S^1 is a compact parallelizable manifold of dimension 1, the collection $\overline{\mathsf{Con}}(S^1) := \{\overline{\mathsf{Con}}(S^1, n)\}_{n\geq 1}$ is, by Theorem 4.5.7, a right module over the operad F_1 which was discussed at great length in Example **??**. This induces an action of the nonsymmetric operad $K = \{K_n\}_{n\geq 1}$ of the associahedra on $W = \{W_n\}_{n\geq 1}$. This action was studied and very explicitly described in **??**. It was also shown in the same paper that the contractibility of the cyclohedra reflects a certain property of Koszulness for modules over operads.

For a general, non-parallelizable manifold, one needs extra data that would identify the tangent space $T_x(M)$ with $\mathbb{R}^k$, for any point $x \in M$. These data are provided by a *framing*, that is, one must consider the framed configuration space $f\mathsf{F}(M, n)$ which we introduced in Definition 4.1.9. The compactification $\mathfrak{f}\overline{\mathsf{Con}}(M) = \{\mathfrak{f}\overline{\mathsf{Con}}(M, n)\}_{n\geq 1}$ is then indeed a right F_k-module. It is, in fact, a module over a bigger operad $\mathfrak{f}\mathsf{F}_k$ which we introduce now.

Framed operad $\mathfrak{f}\mathsf{F}_k$. The natural action of the orthogonal group $O(k)$ on $\mathbb{R}^k$ induces a natural left action on the configuration space $\mathsf{F}(\mathbb{R}^k, n)$ which in turn induces a left action on the quotient $\mathring{\mathsf{F}}_k(n) = \mathsf{F}(\mathbb{R}^k, n)/\mathrm{Aff}$. In terms of normal representatives introduced in **??**, this action is described by

$$(z^1, \dots, z^n) \longmapsto (gz^1, \dots, gz^n),$$

where $(z^1, \dots, z^n)$ is a normal representative of a point of $\mathring{\mathsf{F}}_k(n)$ and $g \in O(n)$. Observe that the configuration $(gz^1, \dots, gz^n)$ is normal, too.

The above action induces a left $O(k)$-action on the compactification $\mathsf{F}_k(n)$ satisfying (4.2), and we define

$$\mathfrak{f}\mathsf{F}_k := \mathsf{F}_k \rtimes O(k), \tag{4.45}$$

the semidirect product introduced in Definition **??**.

We will need for the proof of Theorem 4.5.10 a more detailed description obtained by decorating the steps of the construction of $\mathsf{F}_k(n)$ by orthogonal frames. We use the notation introduced in Section 4.3. Namely, for $n \geq 2$ and $T \in \mathtt{Tree}_n^r$ we introduce

$$\mathfrak{f}Y[T] := Y[T] \times O(k)^{\times n}, \tag{4.46}$$

with elements written as

$$\xi \times (g_1, \dots, g_n),\ \xi \in Y[T],\ (g_1, \dots, g_n) \in O(k)^{\times n}. \tag{4.47}$$

Observe that the collection $\mathfrak{f}Y = \{\mathfrak{f}Y(n)\}_{n \geq 1}$ with

$$\mathfrak{f}Y(n) := \bigsqcup_{T \in \mathtt{Tree}_n^r} \mathfrak{f}Y[T]$$

is an operad. Indeed we already know from the proof of Proposition **??** that the collection $Y = \{Y(n)\}_{n \geq 1}$ is an operad. Each $Y(n)$ clearly admits a natural left action of the orthogonal group $O(k)$ which satisfies (4.2). The operad structure of $\mathfrak{f}Y$ is then given by the identification

$$\mathfrak{f}Y \cong Y \rtimes O(k).$$

Define the *extended set* of composable elements by

$$\mathfrak{f}\widehat{U}[T] := \widehat{U}[T] \times O(k)^{\times n}.$$

Then the framed compactification $\mathfrak{f}\mathsf{F}_k(n)$ can be defined also as

$$\mathfrak{f}\mathsf{F}_k(n) := \mathfrak{f}\widehat{U}(n)/\sim,$$

where

$$\mathfrak{f}\widehat{U}(n) := \bigsqcup_{T\in\mathtt{Tree}^r_n} \mathfrak{f}\widehat{U}[T],$$

and the equivalence $\sim$ is given by

$$\mathfrak{f}\widehat{U}(n) \supset \mathfrak{f}\widehat{U}[T] \ni \xi \times \vec{g} \sim (0,\gamma(\xi)) \times \vec{g} \in \mathfrak{f}\widehat{U}[T_h] \subset \mathfrak{f}\widehat{U}(n), \tag{4.48}$$

for $T \in \mathtt{Tree}^r_n$, $\xi \in \widehat{U}[T]$ and $\vec{g} \in O(k)^{\times n}$, with $\gamma(\xi)$ defined in (**??**).

Compactification of $f\mathsf{F}(M,n)$. The shortest way to define the compactification $\mathfrak{f}\overline{\mathsf{Con}}(M,n)$ of the framed configuration space $f\mathsf{F}(M,n)$ is to say that it is the pullback of the diagram

$$\begin{tikzcd} & O(M)^{\times n} \arrow[d, "\pi_M^{\times n}"] \\ \overline{\mathsf{Con}}(M,n) \arrow[r, "\mathrm{bld}"] & M^{\times n} \end{tikzcd} \tag{4.49}$$

where $\mathrm{bld} : \overline{\mathsf{Con}}(M,n) \to M^{\times n}$ is the blowing-down map (4.42) and $\pi_M : O(M) \to M$ the orthogonal frame bundle of M. The following proposition follows immediately from dProposition 4.5.3, since $\mathfrak{f}\overline{\mathsf{Con}}(M,n)$ is, as the pullback of (4.49), 'locally' the product $\overline{\mathsf{Con}}(M,n) \times O(k)^{\times n}$.

Proposition 4.5.9. *The space $\mathfrak{f}\overline{\mathsf{Con}}(M,n)$ is a smooth manifold-with-corners of dimension*

$$\dim(\overline{\mathsf{Con}}(M,n)) + n\dim(O(k)) = kn + \frac{kn(k-1)}{2} = \frac{kn(k+1)}{2}$$

containing the framed configuration space $f\mathsf{F}(M,n)$ as an open dense subset. It is compact if and only if the manifold M is.

In order to prove that the collection $\mathfrak{f}\overline{\mathsf{Con}}(M) = \{\mathfrak{f}\overline{\mathsf{Con}}(M,n)\}_{n\geq 1}$ is a right $\mathfrak{f}\mathsf{F}_k$-module, one must understand better the structure of these objects. Let $f\mathsf{F}(M)(T)$ be the pullback of the diagram

$$\begin{tikzcd} & O(M)^{\times n} \arrow[d, "\pi_M^{\times n}"] \\ \mathsf{F}(M)(T) \arrow[r, "\mathrm{bld}_T"] & M^n \end{tikzcd}$$

where the map bld_T was defined in (4.43). For $T \in \mathtt{Tree}^e_n$ let

$$\mathfrak{f}Z[T] := \mathrm{Met}(T) \times f\mathsf{F}(M)(T), \tag{4.50}$$

compare (4.13). An element of $\mathfrak{f}\eta \in \mathfrak{f}Z[T]$ is of the form

$$\mathfrak{f}\eta = \eta \times \vec{\mathfrak{f}},\ \vec{\mathfrak{f}} = (\mathfrak{f}_1,\ldots,\mathfrak{f}_n), \tag{4.51}$$

where η is as in (4.32) and $\mathfrak{f}_i$ is a frame at $x^{\mathrm{pa}(i)}$, $1 \leq i \leq n$. Define the composable elements and the extended set by

$$\begin{aligned} \mathfrak{f}W[T] &:= \{\eta \times \vec{\mathfrak{f}} \in \mathfrak{f}W[T];\ \eta \in W[T]\} \text{ and} \\ \mathfrak{f}\widehat{W}[T] &:= \{\eta \times \vec{\mathfrak{f}} \in \mathfrak{f}W[T];\ \eta \in \widehat{W}[T]\} \end{aligned}$$

with $W[T]$ (resp. $\widehat{W}[T]$) as in (4.34) (resp. in (4.36)). Finally, each $\mathfrak{f}\eta \in \mathfrak{f}\widehat{W}[T]$ determines $\delta_{\mathfrak{f}}(\mathfrak{f}\eta) \in f\mathsf{F}(M)(T)$ by the formula

$$\delta_{\mathfrak{f}}(\mathfrak{f}\eta) := \delta(\eta) \times \Phi(\vec{\mathfrak{f}}),$$

where $\delta(\eta) \in \mathsf{F}(M, T)$ was defined in (4.40) and

$$\Phi(\vec{\mathfrak{f}}) := (\Phi_{x^{\mathrm{pa}_T(1)}, y^{\mathrm{pa}_{T_h}(1)}}(\mathfrak{f}_1), \ldots, \Phi_{x^{\mathrm{pa}_T(n)}, y^{\mathrm{pa}_{T_h}(n)}}(\mathfrak{f}_n)),$$

where $\vec{y}$ has the same meaning as in (4.38). The space

$$\mathfrak{f}\overline{\mathsf{Con}}(M, n) := \mathfrak{f}\widetilde{W}(n)/\sim$$

with

$$\mathfrak{f}\widetilde{W}(n) := \bigsqcup_{T \in \mathtt{Tree}_n^e} \mathfrak{f}\widetilde{W}[T]$$

and the relation $\sim$ given by

$$\mathfrak{f}\widehat{W}(n) \supset \mathfrak{f}\widehat{W}[T] \ni \mathfrak{f}\eta \sim (0, \delta_{\mathfrak{f}}(\mathfrak{f}\eta)) \in \mathfrak{f}\widehat{W}[T_h] \subset \mathfrak{f}\widehat{W}(n), \tag{4.52}$$

for $T \in \mathtt{Tree}_n^e$ and $n \geq 1$, then coincides with the pull-back of the diagram (4.49).

Theorem 4.5.10. *Let M be a complete Riemannian manifold. The collection $\mathfrak{f}\overline{\mathsf{Con}}(M) = \{\mathfrak{f}\overline{\mathsf{Con}}(M, n)\}_{n \geq 1}$ is then a natural right $\mathfrak{f}\mathsf{F}_k$-module.*

Proof. The scheme of the proof is similar to that of Proposition **??**. For $m, n \geq 1$ define

$$\mathfrak{f}Z(n) := \bigsqcup_{T \in \mathtt{Tree}_n^e} \mathfrak{f}Z[T]$$

with $\mathfrak{f}Z[T]$ as in (4.50) and

$$\mathfrak{f}Y(m) := \bigsqcup_{S \in \mathtt{Tree}_m^r} \mathfrak{f}Y[S]$$

with $\mathfrak{f}Y[S]$ as in (4.46).

The most important step of the proof is to observe that the collection $\mathfrak{f}Z = \{\mathfrak{f}Z(n)\}_{n \geq 1}$ is a right module over the operad $\mathfrak{f}Y = \{\mathfrak{f}Y(n)\}_{n \geq 1}$. This module structure is defined by grafting underlying trees, using the frames to identify tangent spaces at various points. It is described, in terms of $\circ_i$-operations, as follows.

Let $T \in \mathtt{Tree}^e_n$, $S \in \mathtt{Tree}^r_m$ and $1 \leq i \leq n$. Let $\eta \times (\mathfrak{f}_1, \ldots, \mathfrak{f}_n) \in \mathfrak{f}Z[T] \subset \mathfrak{f}Z(n)$ be as in (4.51) and $\xi \times (g_1, \ldots, g_m) \in \mathfrak{f}Y[S] \subset \mathfrak{f}Y(m)$ as in (4.47). The ith frame $\mathfrak{f}_i$ defines an identification of $T_{x^{\mathrm{pa}(i)}}M$ and $\mathbb{R}^k$. So it also identifies $\xi \in Y[S]$ with a point $\mathfrak{f}_i(\xi) \in Y_{x^{\mathrm{pa}(i)}}[S]$ and $(g_1, \ldots, g_m) \in O(k)^{\times m}$ with a vector $(\mathfrak{f}_i(g_1), \ldots, \mathfrak{f}_i(g_m))$ of frames at $x^{\mathrm{pa}(i)} \in M$. This vector can be of course described, using the natural left action of the orthogonal group on frames, as

$$(\mathfrak{f}_i(g_1), \ldots, \mathfrak{f}_i(g_m)) = (g_1^{-1}(\mathfrak{f}_i), \ldots, g_m^{-1}(\mathfrak{f}_i))$$

We then define

$$\begin{aligned} &[\eta \times (\mathfrak{f}_1, \ldots, \mathfrak{f}_n)] \circ_i [\xi \times (g_1, \ldots, g_m)] := \\ &:= \ \ \mu \times (\mathfrak{f}_1, \ldots, \mathfrak{f}_{i-1}, \mathfrak{f}_i(g_1), \ldots, \mathfrak{f}_i(g_m), \mathfrak{f}_{i+1}, \ldots, \mathfrak{f}_m) \in \mathfrak{f}Z(m+n-1), \end{aligned} \tag{4.53}$$

where μ is the coloring of $T \circ_i S$ given by η on T and ξ on S.

It is evident that the above action restricts to an action of the extended collection $\mathfrak{f}\widehat{U} = \{\mathfrak{f}\widehat{U}(n)\}_{n\geq 1}$ on the extended collection $\mathfrak{f}\widehat{W} = \{\mathfrak{f}\widehat{W}(n)\}_{n\geq 1}$ and that it commutes with the defining relations (4.52) and (4.48). Therefore it induces the requisite action of the quotient spaces. This finishes the proof of the theorem. ■

Remark 4.5.11. The unframed operad F_k is a suboperad of $\mathfrak{f}\mathsf{F}_k$, therefore the collection $\mathfrak{f}\overline{\mathsf{Con}}(M) = \{\mathfrak{f}\overline{\mathsf{Con}}(M, n)\}_{n\geq 1}$ is also a right F_k-module, with the action given by the obvious modification of (4.53):

$$\begin{aligned} &[\eta \times (\mathfrak{f}_1, \ldots, \mathfrak{f}_n)] \circ_i \xi := \\ &:= \ \ \mu \times (\mathfrak{f}_1, \ldots, \mathfrak{f}_{i-1}, \underbrace{\mathfrak{f}_i, \ldots, \mathfrak{f}_i}_{i \text{ times}}, \mathfrak{f}_{i+1}, \ldots, \mathfrak{f}_m) \in \mathfrak{f}Z(m+n-1). \end{aligned} \tag{4.54}$$

Proof of Theorem 4.5.7. If the manifold M is parallelizable, then clearly

$$\mathfrak{f}\overline{\mathsf{Con}}(M)(n) \cong \overline{\mathsf{Con}}(M)(n) \times O(k)^{\times n}, \tag{4.55}$$

with the isomorphism depending on the choice of the trivialization of the tangent bundle of M. The action (4.54) clearly does not affect the second factor of (4.55), therefore it induces an action on the compactification of the unframed configuration space. ■

Chapter 5

Generalization of Operads

5.1. Cyclic operads.

Version of January 17, 1999

One of the first sections written. Needs a deep revision.

In **??** we introduced the operad $\mathcal{M}_0 = \{\mathcal{M}_0(n)\}_{n\geq 1}$ of punctured Riemann spheres. The operadic right Σ_n-action on $\mathcal{M}_0(n)$ permuted the labels $\{1,\ldots,n\}$ of punctures. But each $\mathcal{M}_0(n)$ obviously admits a higher type of symmetry – the natural right Σ_{n+1}-action which permutes the labels $\{0,\ldots,n\}$ of *all* punctures, including the label of the 'output.'

This example indicates that, for some operads, there is no clear distinction between 'inputs' and the 'output.' Cyclic operads, introduced by E. Getzler and M.M. Kapranov, formalize this phenomenon. They are, roughly speaking, operads with an extra symmetry that interchanges the output with the inputs. The material of this section and the following sections is taken mostly from their paper [37].

We interpret the symmetric group Σ_{n+1} as the group of permutations of the set $\{0,\ldots,n\}$ and Σ_n as the subgroup of Σ_{n+1} consisting of permutations $\sigma \in \Sigma_{n+1}$ with $\sigma(0) = 0$. If $\tau_n \in \Sigma_{n+1}$ denotes the cycle $(0,\ldots,n)$, then τ_n and Σ_n generate Σ_{n+1}.

Definition 5.1.1. *A* cyclic operad *is an ordinary operad $\mathcal{P}$ such that the right Σ_n-action on $\mathcal{P}(n)$ extends, for $n \geq 1$, to an action of Σ_{n+1} such that*

(i) $\tau_1(1) = 1$, where $1 \in \mathcal{P}(1)$ is the unit and

(ii) for $p \in \mathcal{P}(m)$ and $q \in \mathcal{P}(n)$ the composition maps satisfy

$$\gamma(p; q, 1, \ldots, 1) \cdot \tau_{m+n-1} = (-1)^{|p|\cdot|q|}\gamma(q \cdot \tau_n; 1, \ldots, 1, p \cdot \tau_m).$$

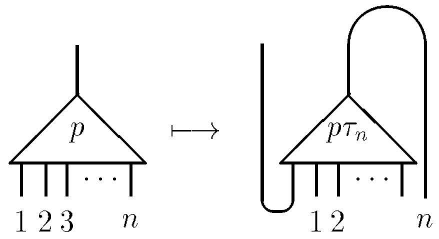


Figure 5.1: A 'visualization' of the action of τ_n. The element τ_n turns $p \in \mathcal{P}(n)$, represented as a 'thing' with n inputs and one output, a bit so that the first input becomes the output and the output becomes the last input of $p \cdot \tau_n$.

An intuitive feeling for the action of τ_n is suggested by Figure 5.1.

We saw in **??** that an (ordinary) operad is an algebra over the triple $\mathbf{T} : \mathbf{Coll} \to \mathbf{Coll}$ of rooted trees. We show that cyclic operads are algebras over a similar triple based on *unrooted* trees. Recall **??** that a collection is a sequence $E = \{E(n)\}_{n \geq 0}$ of vector spaces together with a right action of Σ_n on $E(n)$ for each n. We define a *cyclic collection* to be a sequence $W = \{W(n)\}_{n \geq 0}$ such that each $W(n)$ is a (right) Σ_{n+1}-module. Let $\mathbf{Coll}_+$ denote the category of cyclic collections.

We defined in **??**, for each collection E, a functor $E \longmapsto E(I)$ from the category of finite sets and their bijections to the category of vector spaces. In the same manner, we define, for a cyclic collection W, a functor $W \longmapsto W((I))$ on the category of non-empty finite sets and their bijections as follows. If I is a set with $n+1$ elements, then

$$W((I)) := \left(\bigoplus_{f:\{0,\ldots,n\} \to I} W(n) \right)_{\Sigma_{n+1}}. \tag{5.1}$$

This formula has a similar meaning as (**??**), i.e. the summation is taken over all bijections $f : \{0, \ldots, n\} \to I$, the group Σ_{n+1} acts by $(w_f, \sigma) \to (w_{f\circ\sigma^{-1}})\sigma$ and the subscript Σ_{n+1} denotes the space of coinvariants.

Recall **??** that an (unrooted) tree is a finite contractible graph. We denoted by $\mathrm{Vert}(T)$ the set of vertices of a tree T and, for a vertex $v \in \mathrm{Vert}(T)$, by $\mathrm{edg}(v)$ the set of edges incident with v. For a cyclic collection W we have the following unrooted versions of **??**:

$$W((T)) := \bigotimes_{v \in \mathrm{Vert}(T)} W((\mathrm{edg}(v)))$$

and

$$\mathbf{T}_+ W(n) := \bigoplus_{\text{unrooted } n+1\text{-trees}} W((T)). \tag{5.2}$$

Grafting of trees defines a natural map $\mu_W : \mathbf{T}_+\mathbf{T}_+ W \to \mathbf{T}_+ W$. We also have embeddings $\eta_W : W \to \mathbf{T}_+ W$, which identify $W(n)$ with the summand in (5.2) corresponding to the unique unrooted tree with one vertex and $n+1$ external edges. The following theorem was proved in [37].

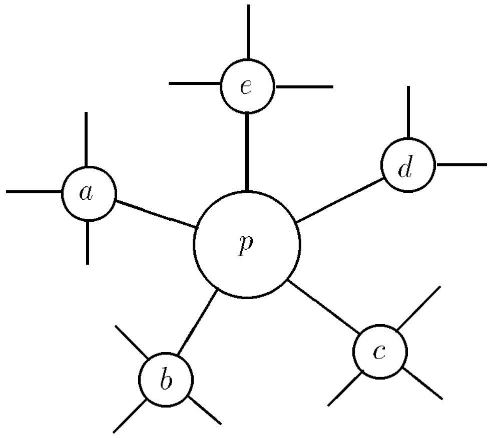
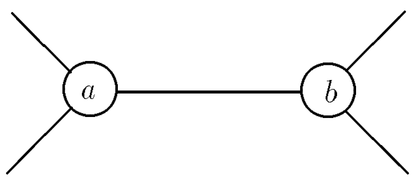


Figure 5.2: The composition of $p \in \mathcal{P}(4)$ with $a, b, c, e \in \mathcal{P}(3)$ and $d \in \mathcal{P}(2)$.

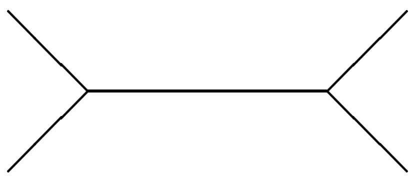
Theorem 5.1.2. *The natural transformations* $\mu : \mathbf{T}_+\mathbf{T}_+ \to \mathbf{T}_+$ *and* $\eta : \mathbb{1} \to \mathbf{T}_+$ *make the functor* $\mathbf{T}_+ : \mathbf{Coll}_+ \to \mathbf{Coll}_+$ *into a triple. An algebra over the triple* $\mathbf{T}_+$ *is the same as a cyclic operad.*

The proof is analogous to the proof of Theorem **??** and we omit it. Recall **??** that, for an ordinary operad $\mathcal{P}$, the space $\mathcal{P}(n)$ is identified with $\mathcal{P}(T)$, where T is the 'n corolla,' i.e. the rooted tree with one vertex and input legs labeled by $1, \ldots, n$. This lead us to the idea of viewing elements of $\mathcal{P}(n)$ as operations with n inputs and one output, and the structure map as the composition of these operations (Figure **??**).

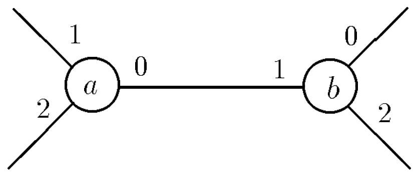
Similarly, for a cyclic operad $\mathcal{P}$, there is an identification of $\mathcal{P}(n)$ and $\mathcal{P}((T))$, where T is the 'n-star,' the unrooted tree with one vertex and $n+1$ input legs indexed by $0, \ldots, n$. This suggests to imagine elements of $\mathcal{P}(n)$ as 'spiders' with $n+1$ legs and the structure map as joining legs of these spiders, see Figure 5.2. But one must be careful when interpreting these pictures. Consider an easier form of this type of picture,



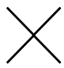
This picture should represent an element of $\mathcal{P}((T))$, where T is the tree



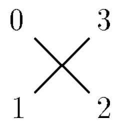
This is, however, not exactly so, unless we choose labels for the edges of $\mathrm{edg}(v)$ for both vertices of T, for example



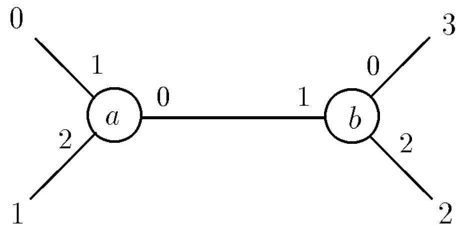
Denote the corresponding element $x(a,b) \in \mathcal{P}((T))$. The composition $\mu_{\mathcal{P}}(x(a,b))$ is an element of $\mathcal{P}((S))$, where

$S =$ 

To identify it with an element of $\mathcal{P}(3)$, we must label the legs of S, for example, as

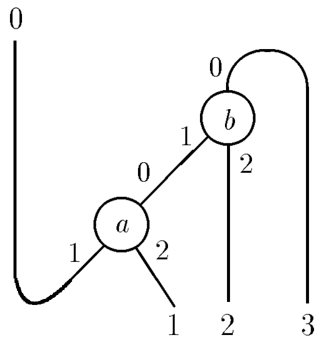


The situation can be summarized as

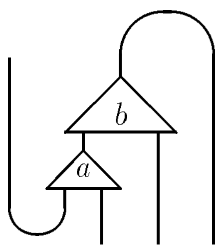


(5.3)

In order to describe the element of $\mathcal{P}(3)$ represented by this picture, redraw it as follows



Now replace the 'spiders' by 'operations' (since a cyclic operad is, after all, an operad) and invoke the standard convention that the inputs are numbered from the left to the right:



The 'visualized' action of Picture 5.1 identifies $\mu_{\mathcal{P}}(x(a,b))$ with $[b(a,1)]\cdot\tau_3$. The last expression uses only the structure data of Definition 5.1.1. An instructive exercise is to try other choices of labels in (5.3).

There is another axiomatization of cyclic operads, close to the 'spider' intuition. Namely, we may say that a cyclic operad is a cyclic collection $\{\mathcal{P}(n)\}_{n\geq 1}$ together with operations

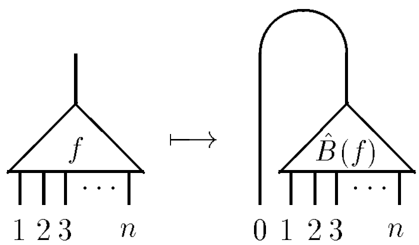
$$ {}_i\circ_j : \mathcal{P}(m)\otimes\mathcal{P}(n)\longrightarrow\mathcal{P}(m+n-1),\ 0\leq i\leq m,\ 0\leq j\leq n, $$

which formalize the 'joining of spider's legs.' These operations have to satisfy certain obvious axioms whose written form is very complicated due to the linear structure of human language.

Example 5.1.3. Let V be a finite-dimensional (graded) vector space and $B : V\otimes V\to\mathbf{k}$ a nondegenerate symmetric bilinear form. The form B induces the identification

$$ \mathit{Hom}V^{\otimes n}V \ni f \longmapsto \hat{B}(f) := B(-,f(-)) \in \mathit{Hom}V^{\otimes(n+1)}\mathbf{k} \tag{5.4} $$

of $\mathit{Hom}V^{\otimes n}V$ and $\mathit{Hom}V^{\otimes(n+1)}\mathbf{k}$. This correspondence can be visualized as follows:

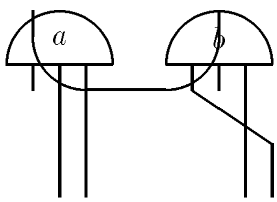


The standard right Σ_{n+1}-action

$$ f\sigma(v_0,\ldots,v_n) = \epsilon(\sigma)\cdot f(v_{\sigma^{-1}(0)},\ldots,v_{\sigma^{-1}(n)}),\ \sigma\in\Sigma_{n+1},\ v_0,\ldots,v_n\in V, $$

where $\epsilon(\sigma)$ is the Koszul sign introduced in **??**, defines, via this identification, a right Σ_{n+1}-action on $\mathit{Hom}V^{\otimes n}V$, that is, on the nth piece of the endomorphism operad $\mathbf{E}_V = \{\mathbf{E}_V(n)\}_{n\geq 1}$. It is easy to show that, with the above action, $\mathbf{E}_V$ is a cyclic operad, called the *cyclic endomorphism operad* on the pair $V=(V,B)$. We will call such pairs *bilinear spaces*.

Cyclic endomorphism operads of Example 5.1.3 suggest another way of viewing elements of the nth piece $\mathcal{P}(n)$ of a cyclic operad, as 'operations' with $n+1$ inputs and no output. The following picture shows how to represent the element $\mu_{\mathcal{P}}(x(a,b))$ of (5.3) in this way.



Consider now a more general situation of a bilinear form $B : A \otimes A \to U$ with values in a (graded) vector space U. Suppose we are given a cyclic operad $\mathcal{P}$ and a $\mathcal{P}$-algebra structure $\alpha : \mathcal{P} \to \mathbf{E}_A$ on A **??**. Define, for each $n \geq 0$, a map $\tilde{B}(\alpha) = \tilde{B}_n(\alpha) : \mathcal{P}(n) \otimes A^{\otimes(n+1)} \to U$ by

$$\tilde{B}(\alpha)(p \otimes a_0 \otimes \cdots \otimes a_n) := \hat{B}(\alpha(p))(a_0, \ldots, a_n), \quad \text{for } a_0, \ldots, a_n \in A,\ p \in \mathcal{P}(n), \tag{5.5}$$

where $\hat{B}(\alpha(p))(a_0, \ldots, a_n) = (-1)^{|a_0||p|} B(a_0, \alpha(p)(a_1, \ldots, a_n))$ was introduced in (5.4).

Definition 5.1.4. *A bilinear form $B : A \otimes A \to U$ is* invariant *(on the $\mathcal{P}$-algebra A) if and only if the maps $\tilde{B}_n(\alpha)$ of (5.5) are, for each $n \geq 0$, invariant under the diagonal action of Σ_{n+1} on $\mathcal{P}(n) \otimes A^{\otimes(n+1)}$.*

Taking, in (5.5), $n = 1$ and $p = 1 \in \mathcal{P}(1)$ we learn that an invariant form is symmetric, $B(a, b) = (-1)^{|a||b|} B(b, a)$, for all $a, b \in A$. We have the following easy proposition.

Proposition 5.1.5. *Let $\mathcal{P}$ be a cyclic operad, $\alpha : \mathcal{P} \to \mathbf{E}_A$ a $\mathcal{P}$-algebra structure on a vector space A, and $B : A \otimes A \to \mathbf{k}$ a nondegenerate symmetric bilinear form. Let us consider $\mathbf{E}_V$ as a cyclic operad, with the cyclic structure given by B as in Example 5.1.3. Then $\alpha : \mathcal{P} \to \mathbf{E}_A$ is a map of cyclic operads if and only if B is invariant in the sense of Definition 5.1.4.*

Proof. Let us analyze when the map $\alpha = \alpha(n) : \mathcal{P}(n) \to \mathbf{E}_A(n)$ is, for $n \geq 1$, Σ_{n+1}-equivariant. Under the identifications of Example 5.1.3 this means that $\hat{B}(\alpha(p)) \cdot \sigma = \hat{B}(\alpha(p \cdot \sigma))$, for $p \in \mathcal{P}(n)$ and $\sigma \in \Sigma_{n+1}$ or, in 'coordinates,'

$$\hat{B}(\alpha(p))(\sigma(a_0, \ldots, a_n)) = \hat{B}(\alpha(p \cdot \sigma))(a_0, \ldots, a_n).$$

But this is clearly the Σ_{n+1}-invariance of Definition 5.1.4. ■

In the situation of Proposition 5.1.5 we say that A is a *cyclic $\mathcal{P}$-algebra*. This technical notion will be useful later in **??**.

Recall **??** that an operad $\mathcal{P}$ is quadratic if it has a presentation $\mathcal{P} = \langle E; R \rangle$ $(= \mathcal{F}(E)/(R))$, where $E = \mathcal{P}(2)$ and $R \subset \mathcal{F}(E)(3)$. The action of Σ_2 on E extends to an action of Σ_3 via the sign representation $\mathrm{sgn} : \Sigma_3 \to \{\pm 1\} = \Sigma_2$. It can be easily verified that this action induces a cyclic operad structure on the free operad $\mathcal{F}(E)$. In particular, $\mathcal{F}(E)(3)$ is a right Σ_4-module. We say that $\mathcal{P}$ is a *cyclic quadratic operad* if, in the above presentation, R is a Σ_4 invariant subspace. In this case, $\mathcal{P}$ has an induced cyclic structure.

Theorem 5.1.6. *Each quadratic operad $\mathcal{P} = \langle E; R \rangle$ with $\dim(E) = 1$ is cyclic quadratic.*

Proof. The representation of Σ_2 on E may be either the trivial representation $\mathbb{1}$ or the signum representation sgn. This means that the operad $\mathcal{P}$ has either a commutative, or an anticommutative product.

Commutative case. A direct calculation shows that

$$\mathcal{F}(E)(3) \cong \mathbb{1} \oplus V_{2,2}$$

as a right Σ_4-module (see **??** for the notation). On the restriction to the subgroup $\Sigma_3 \subset \Sigma_4$ it becomes

$$\mathcal{F}(E)(3) \cong \mathbb{1} \oplus V_{2,1}.$$

Thus the Σ_4- and Σ_3-invariant subspaces of $\mathcal{F}(E)(3)$ are the same, so that each Σ_3-invariant subspace R of $\mathcal{F}(E)(3)$ defines a cyclic quadratic operad.

Anticommutative case. The arguments are similar. We have

$$\mathcal{F}(E)(3) \cong \mathrm{sgn} \oplus V_{2,2}$$

over Σ_4 and

$$\mathcal{F}(E)(3) \cong \mathrm{sgn} \oplus V_{2,1}.$$

as Σ_3-modules. This finishes the proof. ■

An immediate consequence of Theorem 5.1.6 is that the operads *Comm* **??** and *Lie* **??** are cyclic quadratic. A symmetric bilinear form $B : A \otimes A \to U$ on a commutative associative algebra $A = (A, \cdot)$ is invariant if

$$B(a, b \cdot c) = B(a \cdot b, c), \quad \text{for all } a, b, c \in A, \tag{5.6}$$

which is (5.5) for $n = 2$. This follows from an obvious fact that, for quadratic operads, the validity of (5.5) for $n = 1$ (which is the symmetry of B) and $n = 2$ implies (5.5) for an arbitrary n.

Similarly, a bilinear form $B : L \otimes L \to U$ on a Lie algebra $L = (L, [-,-])$ is invariant if and only if

$$B([u, v], w) = B(u, [v, w]) \ \text{ for } u, v, w \in U.$$

The operads *Ass* **??** and *Poiss* **??** are cyclic quadratic. Observe that this fact does not follow from Theorem 5.1.6, since now $\dim(E) = 2$,

$$E = \text{the regular representation of } \Sigma_2 = \mathbb{1} \oplus \mathrm{sgn}, \tag{5.7}$$

but it can be verified directly, by a similar calculation as in the proof of Theorem 5.1.6. Equation (5.7) expresses the elementary but surprising fact that, in characteristic zero, one bilinear operation is the same as one symmetric and one antisymmetric operation.

A symmetric bilinear form $B : A \otimes A \to U$ on an associative algebra $A = (A, \cdot)$ is invariant if $B(a \cdot b, c) = B(a, b \cdot c)$ (formally the same as (5.6)). If A has a unit 1, there is a one-to-one correspondence between such invariant bilinear forms and *traces* on A, that is, linear maps $T : A \to U$ such that $T(a \cdot b) = T(b \cdot a)$, for $a, b \in A$. The correspondence is given by $T(a) = B(1, a)$, resp. $B(a, b) = T(a \cdot b)$.

In [37], a complete classification of cyclic quadratic operads with $\mathcal{P}(2) = 1\!\!1 \oplus \mathrm{sgn}$ is given – there are 80 of those. An example of an operad which is quadratic but not cyclic quadratic is the operad *Leib* for Leibniz algebras, which we introduced in **??**, see [37] for the details.

Example 5.1.7. We defined **??** the topological operad $\mathcal{K} = \{\mathcal{K}(n)\}_{n \geq 1}$ for A_∞-spaces as the symmetrization of the non-Σ topological operad $\underline{\mathcal{K}} = \{K_n\}_{n \geq 1}$, where K_n is the nth Stasheff polyhedron. The vertices of K_n correspond to complete parenthesizations of the string $1 \ldots n$. Such parenthesizations are in bijective correspondence with triangulations of an (n+1)-gon P_{n+1} into triangles whose vertices are among the vertices of P_{n+1}, while faces of K_n correspond to decompositions of P_{n+1} into polygons, see, for example, [55]. Taking P_{n+1} to be a regular polygon, we obtain an action of $\mathbf{Z}_{n+1}$ on P_{n+1} by rotation through multiples of the angle $2\pi/(n+1)$, inducing an action on K_n. Now $\mathcal{K}(n)$ is the induced Σ_{n+1}-space $\mathrm{Ind}_{\mathbf{Z}_{n+1}}^{\Sigma_{n+1}} K_n$. This interpretation shows that each $\mathcal{K}(n)$ admits a natural right Σ_{n+1}-action. This action defines a cyclic operad structure on $\mathcal{K}$.

Example 5.1.8. We already indicated at the beginning of this section that the operad $\mathcal{M}_0$ of genus zero Riemann surfaces with parametrized punctures is a topological cyclic operad. This immediately implies that the operad $BV = H_*(\mathcal{M}_0; \mathbf{k})$ for Batalin-Vilkovisky algebras **??** is a cyclic operad in the category of graded vector spaces.

Like ordinary operads, also cyclic operads have a non-Σ variant. We say that a non-Σ operad $\mathcal{P}$ **??** is a *non-Σ cyclic operad*, if each $\mathcal{P}(n)$ has a right $\mathbf{Z}_{n+1}$-action such that axioms (i) and (ii) of Definition 5.1.1, with τ_n interpreted as an element of $\mathbf{Z}_{n+1}$, are satisfied. The prominent examples of nonsymmetric cyclic operads are the non-Σ operad $\underline{Ass}$ for associative algebras **??**, the non-Σ topological operad $\underline{\mathcal{K}}$ for A_∞-spaces **??**, and the non-Σ operad $\underline{A_\infty}$ for A_∞ algebras **??**. Non-Σ cyclic operads, as simpler versions of cyclic operads, were introduced and studied in [82].

5.2. Application: cyclic (co)homology.

Version of February 10, 1998

The cyclic cohomology of an associative unital algebra was introduced by A. Connes [18] in 1983 as one of basic tools of his noncommutative geometry. In the same year, B.L. Tsygan [127] used the cyclic homology for the computation of homology groups of some Lie algebras; this approach was further developed by J.-L. Loday and D. Quillen in [74].

Let us briefly recall the definitions. For a (graded) unital associative algebra A over $\mathbf{k}$ (which can be now any commutative ring), let $C_n(A) := A^{\otimes(n+1)}$ with the standard grading

$$\deg(a_0 \otimes \cdots \otimes a_n) = \sum_{0 \le i \le n} \deg(a_i) + n.$$

Let us define degree -1 linear operators $b, b' : C_n(A) \to C_{n-1}(A)$ by the formulas

$$\begin{aligned} b'(a_0, \ldots, a_n) &:= \sum_{i=0}^{n-1} (-1)^i (a_0, \ldots, a_i a_{i+1}, \ldots, a_n), \text{ and} \\ b(a_0, \ldots, a_n) &:= b'(a_0, \ldots, a_n) + (-1)^n (a_n, a_0, \ldots, a_{n-1}). \end{aligned}$$

The complex $(C_*(A), b')$ is the standard resolution of the algebra A as an $A \otimes A^{\mathrm{op}}$-module, hence it is acyclic. The complex $CH_*(A; A) := (C_*(A), b)$ is the complex calculating the Hochschild homology $HH_*(A; A)$ of A with coefficients in itself,

$$H_*(C_*(A), b) := \mathrm{Tor}_*^{A \otimes A^{\mathrm{op}}}(A, A).$$

The cyclic group $\mathbb{Z}_{n+1}$ acts on $C_n(A)$ by $t(a_0, \ldots, a_n) := (-1)^n (a_n, a_0, \ldots, a_{n-1})$, where t is the generator. Let $N := 1 + t + \cdots + t^n \in \mathbf{k}[\mathbb{Z}_{n+1}]$ be the norm operator in the group algebra. A crucial fact is that

$$b(1-t) = (1-t)b' \text{ and } b'N = Nb. \tag{5.8}$$

Thus it makes sense to define the *cyclic homology* $H_*^\lambda(A)$ of A as the homology of the total complex of the bicomplex:

$$\begin{tikzcd} {} \arrow[d, "b"] & {} \arrow[d, "-b'"] & {} \arrow[d, "b"] & \\ A^{\otimes 2} & A^{\otimes 2} \arrow[l, "1-t"'] & A^{\otimes 2} \arrow[l, "N"'] & {} \arrow[l, "1-t"'] \\ {} \arrow[d, "b"] & {} \arrow[d, "-b'"] & {} \arrow[d, "b"] & \\ A & A \arrow[l, "1-t"'] & A \arrow[l, "N"'] & {} \arrow[l, "1-t"'] \end{tikzcd} \tag{5.9}$$

Let us assume from now on that the characteristic of $\mathbf{k}$ is zero. Under this assumption, the acyclicity of rows implies that the cyclic homology can be calculated as the homology of the complex

$$C_*^\lambda(A) := C_*(A)/(1-t)$$

with the differential induced by b (and denoted by the same symbol), $H^\lambda_*(A) = H_*(C^\lambda_*(A), b)$.

The cyclic homology has many miraculous properties. For example, let $\mathfrak{gl}_r(A)$ be the Lie algebra of $r \times r$ matrices with coefficients in A, with the commutator bracket. Denote by $\mathfrak{gl}(A) := \varinjlim \mathfrak{gl}_r(A)$ the direct limit of the inclusions $\mathfrak{gl}_r(A) \hookrightarrow \mathfrak{gl}_{r+1}(A)$ given, for $r \geq 1$, by

$$\mathfrak{gl}_r(A) \ni M \longmapsto \begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix} \in \mathfrak{gl}_{r+1}(A).$$

Then the space of primitive elements of the Chevalley-Eilenberg homology $H^{\mathrm{CE}}_*(\mathfrak{gl}(A)) := \mathrm{Tor}^{\mathcal{U}(\mathfrak{gl}(A))}(\mathbf{k}, \mathbf{k})$ of $\mathfrak{gl}(A)$ is isomorphic to the shifted cyclic homology of A,

$$\mathrm{Prim}\ H_*(\mathfrak{gl}(A)) \cong H^\lambda_{*-1}(A),$$

see [74, Theorem 6.2]. This should be compared with the isomorphism of the primitives of the Hochschild homology of the infinite matrix general linear group of A and the rationalized algebraic K-theory of A [102],

$$\mathrm{Prim}\ H_*(\mathrm{GL}(A)) \cong K_*(A) \otimes \mathbb{Q}.$$

This explains why the cyclic cohomology is sometimes called also the *additive algebraic K-theory*.

Another interesting property is the relation with the *de Rham cohomology* of a smooth algebra. Recall that, for a commutative unital $\mathbf{k}$-algebra A, the *module of differentials* Ω^1_A is the left A-module defined as the quotient of $A \otimes A$ by the left A-submodule generated by $\{1 \otimes xy - x \otimes y - y \otimes x |\ x, y \in A\}$,

$$\Omega^1_A := (A \otimes A)/\langle 1 \otimes xy - x \otimes y - y \otimes x \rangle. \tag{5.10}$$

It is characterized by the property that, for any left A-module M,

$$\mathrm{Der}_{\mathbf{k}}(A, M) \cong \mathrm{Hom}_A(\Omega^1_A, M),$$

where $\mathrm{Der}_{\mathbf{k}}(A, M)$ is the linear space of derivations of the algebra A with values in M, and $\mathrm{Hom}_A(\Omega^1_A, M)$ is the space of A-linear maps from Ω^1_A to M. A direct verification shows that in fact Ω^1_A coincides with the first Hochschild homology group of the algebra A, $\Omega^1_A \cong HH_1(A; A)$.

The map $d : A \to \Omega^1_A$ given, in the description of (??), by $d(a) := 1 \otimes a$, is a derivation that corresponds to the identity map in $\mathrm{Hom}_A(\Omega^1_A, \Omega^1_A)$. It is called the *universal differential* and extends to a degree 1 differential (denoted again by d) on the exterior A-linear algebra $\Omega^*_A := \bigwedge_A(\Omega^1_A)$ on Ω^1_A. The cohomology $H^*_{\mathrm{DR}}(A) := H^*(\Omega^*_A, d)$ is called the de Rham cohomology of A, see [12, 2.10].

The de Rham cohomology is related to the cyclic homology of a smooth algebra A in the sense of Grothendieck (a prominent example of this type of algebra is the coordinate ring of a smooth variety) by the formula [74, Theorem 2.9]

$$H_n^\lambda(A) \cong \Omega_A^n/d\Omega_A^{n-1} \oplus H_{\mathrm{DR}}^{n-2}(A) \oplus H_{\mathrm{DR}}^{n-4}(A) \oplus \cdots$$

which holds for any $n \geq 0$. An interested reader may find all this and much more in the overview [14].

Technically, the existence of the cyclic homology of an associative algebra is a consequence of the stunning fact that the action of the cyclic group on $C_*(A)$ behaves well with respect to the differentials, as expressed by (5.8). E. Getzler and M.M. Kapranov realized that this follows from the cyclicity of the operad Ass which governs associative algebras. They then defined the cyclic homology for an arbitrary category of algebras over a cyclic operad. Their definition is based on the notion of a *universal bilinear invariant form*.

Recall (Definition 5.1.4) that, given a $\mathcal{P}$-algebra A, $\alpha : \mathcal{P} \to \mathit{End}_A$, over a cyclic operad $\mathcal{P}$, a bilinear form $B : A \otimes A \to U$ is said to be invariant if and only if, for any $n \geq 1$, $\sigma \in \Sigma_{n+1}$, $a_0, \ldots, a_1 \in A$ and $p \in \mathcal{P}(n)$,

$$(-1)^{|a_0||p|} B(a_0, \alpha(p)(a_1, \ldots, a_n)) = (-1)^{|a_{\sigma(0)}||p|} \epsilon(\sigma) B(a_{\sigma(0)}, \alpha(p)(a_{\sigma(1)}, \ldots, a_{\sigma(n)})), \qquad (5.11)$$

where $\epsilon(\sigma) = \epsilon(\sigma; a_0, \ldots, a_n)$ is the Koszul sign **??** of the permutation σ.

Definition 5.2.1. *Let A be an algebra over a cyclic operad $\mathcal{P}$, $\alpha : \mathcal{P} \to \mathit{End}_A$. We say that an invariant bilinear form $T : A \otimes A \to \Lambda$ with values in a vector space Λ is* universal *if, for any invariant bilinear form $B : A \otimes A \to U$ there exist a unique linear map $\omega : \Lambda \to U$ such that $B = \omega \circ T$.*

The universal bilinear invariant form for $\mathcal{P}$ can be constructed as follows. Take $\Lambda = \Lambda(\mathcal{P}, A)$ to be the coequalizer of the maps

$$\mathcal{P}(n) \otimes A^{\otimes(n+1)} \xrightarrow{\sigma} \mathcal{P}(n) \otimes A^{\otimes(n+1)} \xrightarrow{R_n} A \otimes A, \ n \geq 1, \ \sigma \in \Sigma_{n+1},$$

where $R_n(p \otimes a_0 \otimes \cdots \otimes a_n) := (-1)^{|a_0||p|} a_0 \otimes \alpha(p)(a_1, \ldots, a_n)$ and σ acts diagonally. In other words, $\Lambda(\mathcal{P}, A)$ is the quotient of the product $A \otimes A$ modulo the identifications

$$(-1)^{|a_0||p|} a_0 \otimes \alpha(p)(a_1, \ldots, a_n) \sim (-1)^{|a_{\sigma(0)}||p|} \epsilon(\sigma) a_{\sigma(0)} \otimes \alpha(p)(a_{\sigma(1)}, \ldots, a_{\sigma(n)})$$

required by (5.30). Then we define $T : A \otimes A \to \Lambda$ to be the canonical projection. The universal property of the map $T : A \otimes A \to \Lambda(\mathcal{P}, A)$ is immediately obvious. For $\mathcal{P} = \mathit{Ass}$, Comm and Lie, the space $\Lambda(\mathcal{P}, A)$ was introduced by M. Kontsevich [62] who denoted it $\Omega^0(A)$.

Example 5.2.2. If $\mathcal{P} = \mathit{Ass}$ and $A = (A, \cdot)$ is a unital associative algebra, then $\Lambda = A/[A, A]$ is the abelianization A_{ab} of A and $T(a \otimes b) \in \Lambda$ is the equivalence class of $a \cdot b$ in A_{ab}.

If $\mathcal{P} = \mathit{Comm}$ and $C = (C, \cdot)$ a commutative unital algebra, then $\Lambda = C$ and $T : C \otimes C \to C$ is the multiplication.

If $\mathcal{P} = \mathit{Poiss}$ and $P = (P, \{-,-\}, \cdot)$ is a unital Poisson algebra (this means that there exist a unit $1 \in P$ for the multiplication such that $\{1, P\} = 0$), then $\Lambda(\mathit{Poiss}, P) = P/\{P, P\}$ is the quotient of P by the span of all Poisson brackets.

For a given cyclic operad $\mathcal{P}$, the correspondence $A \longmapsto \Lambda(P, A)$ defines a functor from the category $\mathcal{P}$-dgAlg of differential graded $\mathcal{P}$-algebras to the category of differential graded vector spaces. The category $\mathcal{P}$-dgAlg is a Quillen model category [100] and the functor $\Lambda(\mathcal{P}, -)$ preserves weak equivalences between cofibrations – a quite nontrivial fact proved in [37, Theorem 5.3]. Thus the following definition, due to E. Getzler and M.M. Kapranov, makes sense.

Definition 5.2.3. *Let A be an algebra over a cyclic operad $\mathcal{P}$. The i*th cyclic homology group *$HA_i(A)$ of A is the ith left non-abelian derived functor of the functor $\Lambda(\mathcal{P}, -)$, $HA_i(A) := H_i(\Lambda(\mathcal{P}, F_*))$, where F_* is a cofibrant model of the algebra A.*

Since the theory of non-abelian derived functors would go far beyond the scope of the book, we will not discuss the cyclic homology in the sense of this general definition, but only for algebras over quadratic Koszul operads **??**, in which case we define the cyclic cohomology as the cohomology of a very explicit complex.

So, let $\mathcal{P} = \langle E; R\rangle$ be a cyclic quadratic operad. Then also the quadratic dual **??** $\mathcal{P}^!$ of $\mathcal{P}$ is a cyclic quadratic operad, thus $\mathcal{P}^\perp := \mathbf{s}(\mathcal{P}^!)^*$, the operadic suspension **??** of the linear dual $(\mathcal{P}^!)^*$ of the operad $\mathcal{P}^!$, is an anticyclic cooperad **??**. As a collection,

$$\mathcal{P}^\perp(n) \cong \mathrm{sgn}_{n+1} \uparrow^{n-1} (\mathcal{P}^!(n))^*, \ n \geq 1. \tag{5.12}$$

Let $\mathbf{k}^{n+1}$ denote the standard permutation representation of Σ_{n+1}, i.e. the vector space of $(n+1)$-tuples $(k_0, \ldots, k_n)$ of elements of the ground field $\mathbf{k}$, with the left action of Σ_{n+1} given by

$$\sigma \cdot (k_0, \ldots, k_n) = (k_{\sigma^{-1}(0)}, \ldots, k_{\sigma^{-1}(n)}).$$

Then the n-dimensional representation $V_{n,1}$ of Σ_{n+1} can be described as

$$V_{n,1} = \{(k_0, \ldots, k_n) \in \mathbf{k}^{n+1};\ k_0 + \cdots + k_n = 0\},$$

so there is an exact sequence

$$0 \longrightarrow V_{n,1} \longrightarrow \mathbf{k}^{n+1} \longrightarrow 1\!\!1 \longrightarrow 0 \tag{5.13}$$

of Σ_{n+1}-modules. Combining these data, we are going to define, for any $\mathcal{P}$-algebra A, a natural short exact sequence of chain complexes

$$0 \longrightarrow CA_*(A) \longrightarrow CB_*(A) \longrightarrow \mathcal{C}_*(A) \longrightarrow 0. \tag{5.14}$$

The underlying graded vector spaces of these complexes are as follows:

$$\begin{aligned}
CA_*(A) &:= \bigoplus_{n\geq 1} \mathcal{P}^\perp(n) \otimes_{\Sigma_{n+1}} (V_{n,1} \otimes A^{\otimes(n+1)}),\\
CB_*(A) &:= \bigoplus_{n\geq 1} \mathcal{P}^\perp(n) \otimes_{\Sigma_{n+1}} (\mathbf{k}^{n+1} \otimes A^{\otimes(n+1)}),\\
\mathcal{C}_*(A) &:= \bigoplus_{n\geq 1} \mathcal{P}^\perp(n) \otimes_{\Sigma_{n+1}} (\mathbb{1} \otimes A^{\otimes(n+1)}) \cong \bigoplus_{n\geq 1} \mathcal{P}^\perp(n) \otimes_{\Sigma_{n+1}} A^{\otimes(n+1)},
\end{aligned}$$

with Σ_{n+1} acting diagonally on the factors in parentheses on the right. The maps in (5.14) are induced by (5.13). We are going to define a differential δ on $CB_*(A)$ which preserves $CA_*(A)$ and projects to a differential on $\mathcal{C}_*(A)$.

For any quadratic operad $\mathcal{P} = \langle E; R\rangle$, there is a 'quadratic twisting cochain,' i.e. the degree -1 map $\Psi : \mathcal{P}^\perp \to \mathcal{P}$ of collections given by

$$\Psi(2) : \mathcal{P}^\perp(2) = \uparrow E \xrightarrow{\downarrow} E = \mathcal{P}(2),$$

and $\Psi(n) = 0$, for $n \neq 2$.

Recall **??** that the structure maps of an (anti)cyclic cooperad $\mathcal{Q}$ define, for each finite set K, a map

$$\mathcal{Q}((K)) \to \bigoplus_{K = L \sqcup M} \mathcal{Q}((L^+)) \otimes \mathcal{Q}((M^+)),$$

where $L^+ := L \sqcup \{M\}$ and $M^+ := M \sqcup \{L\}$. In particular, the cocomposition of the anticyclic cooperad $\mathcal{P}^\perp$ gives, for any decomposition $L \sqcup M = \{0, \ldots, n\}$, a map

$$\mathcal{P}^\perp(n) \longrightarrow \mathcal{P}^\perp((L^+)) \otimes \mathcal{P}^\perp((M^+))$$

which induces a map

$$\mathcal{P}^\perp(n) \otimes \mathbf{k}^{n+1} \otimes A^{\otimes(n+1)} \longrightarrow (\mathcal{P}^\perp((L^+)) \otimes A^{\otimes L}) \otimes (\mathcal{P}^\perp((M^+)) \otimes A^{\otimes M}) \otimes \mathbf{k}^{n+1}. \tag{5.15}$$

In the above formula, we also moved the factor $\mathbf{k}^{n+1}$ to the right. This will formally simplify definitions of the maps below. Let us remark that $A^{\otimes L}$ and $A^{\otimes M}$ are unordered tensor products which we introduced in **??**. On the other hand, the twisting cochain Ψ induces a map

$$\Psi : \mathcal{P}^\perp((L^+)) \to \mathcal{P}((L^+))$$

(which, by definition, is nonzero only for $\mathrm{card}(L) = 2$). This map, when composed with the structure map of the $\mathcal{P}$-algebra A,

$$\mathcal{P}((L^+))\otimes A^{\otimes L} \cong \mathcal{P}(L)\otimes A^{\otimes L} \longrightarrow A,$$

gives a map

$$\mathcal{P}^{\perp}((L^+))\otimes A^{\otimes L} \longrightarrow A$$

whose composition with (5.15) gives

$$\mathcal{P}^{\perp}(n)\otimes \mathbf{k}^{n+1}\otimes A^{\otimes(n+1)} \longrightarrow A\otimes\mathcal{P}^{\perp}((M^+))\otimes A^{\otimes M}\otimes\mathbf{k}^{n+1}.$$

Moreover,

$$A\otimes\mathcal{P}^{\perp}((M^+))\otimes A^{\otimes M}\otimes\mathbf{k}^{n+1} \cong \mathcal{P}^{\perp}(M)\otimes A^{\otimes M^+}\otimes\mathbf{k}^{n+1}.$$

The 'summation' $\Sigma : \mathbf{k}^L \to \mathbf{k}$ defined by $\Sigma(\oplus_{l\in L} k_l) := \sum_{l\in L} k_l$ induces the map

$$\mathbf{k}^{n+1} \cong \mathbf{k}^L \oplus \mathbf{k}^M \xrightarrow{\Sigma\oplus 1\!\!1} \mathbf{k}\oplus\mathbf{k}^M \cong \mathbf{k}^{M^+}.$$

Composing all these maps, we obtain a map

$$\delta_M : \mathcal{P}^{\perp}(n)\otimes\mathbf{k}^{n+1}\otimes A^{\otimes(n+1)} \to \mathcal{P}^{\perp}(M)\otimes A^{\otimes M^+}\otimes\mathbf{k}^{M^+}, \tag{5.16}$$

which is in fact nonzero only for $\mathrm{card}(M) = n-1$. Let us denote by $\tilde{\delta}_I$ the sum

$$\mathcal{P}^{\perp}(n)\otimes\mathbf{k}^{n+1}\otimes A^{\otimes(n+1)} \xrightarrow{\sum_M \delta_M} \bigoplus_{\mathrm{card}(M)=n-1} \mathcal{P}^{\perp}(M)\otimes A^{\otimes M^+}\otimes\mathbf{k}^{M^+}. \tag{5.17}$$

The group Σ_{n+1} acts on both sides of (5.17), and $\tilde{\delta}_I$ is Σ_{n+1}-equivariant. Thus, it descends to a map (denoted again by $\tilde{\delta}_I$)

$$\tilde{\delta}_I : \mathcal{P}^{\perp}(n)\otimes_{\Sigma_{n+1}}(\mathbf{k}^{n+1}\otimes A^{\otimes(n+1)}) \longrightarrow \left(\bigoplus_{\mathrm{card}(M)=n-1} \mathcal{P}^{\perp}(M)\otimes A^{\otimes M^+}\otimes\mathbf{k}^{M^+}\right)_{\Sigma_{n+1}}. \tag{5.18}$$

On the other hand, every subset M of $\{0,\ldots,n\}$ is naturally ordered, thus there is a specified order $M \cong \{1,\ldots,n\}$ for any M in the summation above, which means that there are identifications

$$\mathcal{P}^{\perp}(M)\otimes A^{\otimes M^+}\otimes\mathbf{k}^{M^+} \cong \mathcal{P}^{\perp}(n-1)\otimes(\mathbf{k}^n\otimes A^{\otimes n}).$$

These identifications induce the projection

$$\pi : \left(\bigoplus_{\mathrm{card}(M)=n-1} \mathcal{P}^{\perp}(M)\otimes A^{\otimes M^+}\otimes\mathbf{k}^{M^+}\right)_{\Sigma_{n+1}} \longrightarrow \mathcal{P}^{\perp}(n-1)\otimes_{\Sigma_n}(\mathbf{k}^n\otimes A^{\otimes n}).$$

Composing π with the map $\tilde{\delta}_I$ of (5.18), we finally obtain a map δ_I (I from 'inner'),

$$\delta_I : CB_n(A) = \mathcal{P}^\perp(n)\otimes_{\Sigma_{n+1}}(\mathbf{k}^{n+1}\otimes A^{\otimes(n+1)}) \to \mathcal{P}^\perp(n-1)\otimes_{\Sigma_n}(\mathbf{k}^n\otimes A^{\otimes n}) = CB_{n-1}(A).$$

If $A = (A, d)$ is a *differential* graded algebra, then the differential d of A induces another differential δ_A on the complex $CB_*(A)$, by the standard formula

$$\begin{aligned}\delta_A(p\otimes_{\Sigma_{n+1}}((k_0 \oplus \cdots \oplus k_n)\otimes(a_0 \otimes \cdots \otimes a_n))) &:= \\ := \sum_{0\leq i\leq n} (-1)^{(|a_0|+\cdots+|a_{i-1}|+|p|)} p\otimes_{\Sigma_{n+1}}((k_0 \oplus \cdots \oplus k_n)&\otimes(a_0\otimes \cdots \otimes d(a_i)\otimes \cdots \otimes a_n)),\end{aligned}$$

for $p\otimes_{\Sigma_{n+1}}((k_0 \oplus \cdots \oplus k_n)\otimes(a_0 \otimes \cdots \otimes a_n)) \in CB_n(A)$. The following lemma, which is a special case of [37, Lemma 5.11] for quadratic twisting cochains, can be verified directly.

Lemma 5.2.4. *The endomorphism $\delta := \delta_I + \delta_A$ is a differential on $CB_*(A)$. It preserves $CA_*(A)$ and projects to a differential (denoted again by δ) on $\mathcal{C}_*(A)$.*

We can now define the cyclic homology $HA_*(A)$ of A as the homology of the complex (CA_*, δ). The following theorem, which was proved in [37], shows that this definition is consistent with Definition 5.2.3.

Proposition 5.2.5. *Let A be an algebra over a cyclic operad $\mathcal{P}$. Suppose $\mathcal{P}$ is cyclic quadratic and Koszul. Then the homology of the complex $(CA_*(-), \delta)$ calculates the left derived functor of $\Lambda(\mathcal{P}, -)$.*

If we denote the homology of the remaining two complexes in (5.14) by $HB_*(A)$ and $HC_*(A)$, respectively, then we have the long exact sequence

$$\cdots \longrightarrow HC_{n+1}(A)\longrightarrow HA_n(A)\longrightarrow HB_n(A)\longrightarrow HC_n(A)\longrightarrow HA_{n-1}(A)\longrightarrow \cdots \tag{5.19}$$

Example 5.2.6. Let us identify the complexes of (5.14) for the associative algebra case, $\mathcal{P} = Ass$. Recall that the operad Ass is self-dual, $Ass^! = Ass$, and that, as a Σ_{n+1}-module, $Ass(n) = \mathrm{Ind}_{\mathbb{Z}_{n+1}}^{\Sigma_{n+1}}(\mathbb{1})$ **??**, where $\mathbb{1}$ is the one-dimensional trivial representation of $\mathbb{Z}_{n+1}$ and $\mathrm{Ind}_{\mathbb{Z}_{n+1}}^{\Sigma_{n+1}}(\mathbb{1}) := \mathbb{1} \otimes_{\mathbb{Z}_{n+1}} \mathbf{k}[\Sigma_{n+1}]$ is the representation induced along $\mathbb{Z}_{n+1} \hookrightarrow \Sigma_{n+1}$. By (5.12) this means that

$$Ass^\perp(n) = \mathrm{sgn}_{n+1}\uparrow^{n-1} \mathrm{Ind}_{\mathbb{Z}_{n+1}}^{\Sigma_{n+1}}(\mathbb{1}) \cong \uparrow^{n-1} \mathrm{Ind}_{\mathbb{Z}_{n+1}}^{\Sigma_{n+1}}(\mathbb{1}).$$

It follows from the definition of the induced representation that for any left Σ_{n+1}-module M

$$\mathrm{Ind}_{\mathbb{Z}_{n+1}}^{\Sigma_{n+1}}(\mathbb{1})\otimes_{\Sigma_{n+1}} M \cong \mathbb{1}\otimes_{\mathbb{Z}_{n+1}} M \cong M_{\mathbb{Z}_{n+1}} \cong M/(1-t),$$

where $t \in \mathbb{Z}_{n+1}$ is the generator. This means that sequence (5.14) becomes, for $\mathcal{P} = Ass$, the sequence

$$0 \longrightarrow \uparrow^{n-1} (V_{n,1} \otimes A^{\otimes(n+1)})/(1-t) \longrightarrow \uparrow^{n-1} (\mathbf{k}^{n+1} \otimes A^{\otimes(n+1)})/(1-t) \longrightarrow$$
$$\longrightarrow \uparrow^{n-1} A^{\otimes(n+1)}/(1-t) \longrightarrow 0.$$

The map $\alpha : \mathbf{k}^{n+1} \otimes A^{\otimes(n+1)} \longrightarrow A^{\otimes(n+1)}$ given by

$$\alpha((k_0 \oplus \cdots \oplus k_n) \otimes (a_0 \otimes \cdots \otimes a_n)) := (k_0 + k_1 t^{-1} + \cdots + k_n t^{-n})(a_0 \otimes \cdots \otimes a_n)$$

is invariant under the diagonal action of t:

$$\begin{aligned} \alpha\left(t((k_0 \oplus \cdots \oplus k_n) \otimes (a_0 \otimes \cdots \otimes a_n))\right) &= ((k_0 + k_1 t^{-1} + \cdots + k_n t^{-n}) t^{-1})(t(a_0 \otimes \cdots \otimes a_n)) \\ &= (k_0 + k_1 t^{-1} + \cdots + k_n t^{-n})(a_0 \otimes \cdots \otimes a_n) \\ &= \alpha((k_0 \oplus \cdots \oplus k_n) \otimes (a_0 \otimes \cdots \otimes a_n)), \end{aligned}$$

thus it descends to an isomorphism $(\mathbf{k}^{n+1} \otimes A^{\otimes(n+1)})/(1-t) \cong A^{\otimes n+1}$ and converts the above sequence to

$$0 \longrightarrow \uparrow^{n-1} (1-t) A^{\otimes(n+1)} \xrightarrow{\iota} \uparrow^{n-1} A^{\otimes(n+1)} \xrightarrow{\pi} \uparrow^{n-1} A^{\otimes(n+1)}/(1-t) \longrightarrow 0, \qquad (5.20)$$

where ι is the inclusion and π the projection. If one goes through the definition of δ on the middle term of (5.20), one finds that it may be identified with the differential b of the shifted Hochschild complex $CH_*(A, A)$, thus

$$(CB_*(A), \delta) = (\downarrow C_*(A), b).$$

It is also immediately clear from this that the right term of (5.20) is isomorphic to the shifted complex $C^\lambda_*(A)$ calculating the cyclic homology of A, $\mathcal{C}_*(A) = C^\lambda_{*+1}(A)$, therefore $HC_n(A) \cong H^\lambda_{n+1}(A)$. To identify the left term of (5.20), consider bicomplex (5.9) as an exact sequence of chain complexes

$$(C_*(A), b) \xleftarrow{(1-t)} (C_*(A), -b') \xleftarrow{N} (C_*(A), b) \xleftarrow{(1-t)} (C_*(A), -b') \xleftarrow{N} \cdots. \qquad (5.21)$$

Consider also the short exact sequence

$$0 \longrightarrow \mathrm{Ker}(1-t) \hookrightarrow (C_*(A), -b') \xrightarrow{(1-t)} \mathrm{Im}(1-\mathrm{t}) \longrightarrow 0. \qquad (5.22)$$

The exactness of (5.21) gives the isomorphisms

$$\mathrm{Ker}(1-t) \cong \mathrm{Im}(N) \cong \mathrm{Coker}(1-t) = C_*(A)/(1-t)$$

which enables us to identify the sequence (5.22) with

$$0 \longrightarrow (C^\lambda_*(A), b) \xrightarrow{N} (C_*(A), -b') \longrightarrow \uparrow CA_*(A) \longrightarrow 0. \qquad (5.23)$$

Proposition 5.2.7. *If the algebra A is unital, then the complex $CA_*(A)$ is weakly equivalent to $C_*^\lambda(A)$.*

Proof. The proposition follows immediately from the short exact sequence (5.23) and the contractibility of the bar complex $(C_*(A), b')$. ■

To summarize, for an unital associative algebra A, both complexes $CA_*(A)$ and $\mathcal{C}_*(A)$ calculate the cyclic homology, while $CB(A)_*$ calculates the Hochschild homology. The long exact sequence (5.19) can be then identified with the *Connes-Tsygan periodicity exact sequence*

$$\cdots \longrightarrow H_{n+2}^\lambda(A) \xrightarrow{S} H_n^\lambda(A) \xrightarrow{B} H_{n+1}(A, A) \xrightarrow{I} H_{n+1}^\lambda(A) \longrightarrow \cdots$$

where the boundary map S is an additive analog of the Bott periodicity map. This sequence can also be interpreted as a consequence of the periodicity of the bicomplex (5.9). More precisely, observe that if we delete the first two columns of (5.9), we obtain an isomorphic complex, while the cohomology of the quotient is the Hochschild cohomology. These remarks also imply the sequence above.

Example 5.2.8. Consider the Lie algebra case. Since, for any $n \geq 1$, $\mathit{Lie}^!(n) = \mathit{Comm}(n) = \mathbb{1}$, the one-dimensional trivial representation of Σ_{n+1}, we obtain $\mathit{Lie}^\perp(n) \cong \uparrow^{n-1} \mathrm{sgn}_{n+1}$, the shifted signum representation of Σ_{n+1}. By similar methods to those in Example 5.2.6 we infer that, for any Lie algebra $\mathfrak{g}$,

$$CB_n(\mathfrak{g}) = \mathfrak{g} \otimes \uparrow^{n-1} \wedge^n \mathfrak{g}.$$

Thus $CB_*(\mathfrak{g})$ is identified with the shifted Chevalley-Eilenberg chain complex of the Lie algebra $\mathfrak{g}$ with coefficients in itself [47, Chapter VII], $CB_*(\mathfrak{g}) \cong \downarrow C_n(\mathfrak{g}, \mathfrak{g})$. Similarly,

$$\mathcal{C}_*(\mathfrak{g}) \cong \uparrow^{n-1} \wedge^{n+1} \mathfrak{g} = \downarrow^2 C_*(\mathfrak{g}, \mathbf{k}),$$

the shifted Chevalley-Eilenberg chain complex of $\mathfrak{g}$ with trivial coefficients. As a consequence, $CA_*(\mathfrak{g})$ is the kernel of the map $m : C_{*+1}(\mathfrak{g}, \mathfrak{g}) \to C_{*+2}(\mathfrak{g}, \mathbf{k})$ given by the formula

$$m(g_0 \otimes (g_1 \wedge \cdots \wedge g_n)) := g_0 \wedge g_1 \wedge \cdots \wedge g_n.$$

The map m was studied by T. Pirashvili in [98]. He denoted

$$CR_n(\mathfrak{g}) := \mathrm{Ker}(m : C_{n+1}(\mathfrak{g}, \mathfrak{g}) \to C_{n+2}(\mathfrak{g}, \mathbf{k}))$$

and $HR_*(\mathfrak{g})$ the corresponding homology. The dimension shift is chosen because the map m is an isomorphism in degree 0, thus, with this degree convention, $CR_0(\mathfrak{g})$ is the first nontrivial piece of the complex. We may thus identify $CA_*(\mathfrak{g}) = CR_*(\mathfrak{g})$. The associated long exact sequence

$$\cdots \longrightarrow HR_{n-1}(\mathfrak{g}) \longrightarrow H_n^{\mathrm{CE}}(\mathfrak{g}, \mathfrak{g}) \longrightarrow H_{n+1}^{\mathrm{CE}}(\mathfrak{g}, \mathbf{k}) \longrightarrow HR_n(\mathfrak{g}) \longrightarrow \cdots$$

is that of [98, Proposition 1.2].

Example 5.2.9. Let us briefly mention also the case of commutative associative algebras. It can be shown that the complex $(CA_*(C), \delta)$ calculates the shifted Harrison homology of C with trivial coefficients,

$$HA_*(C) \cong \mathrm{Harr}_{*+1}(C, \mathbf{k})$$

and that

$$HC_*(C) := \mathrm{Harr}_{*+1}(C, C),$$

the Harrison cohomology of C with coefficients in itself. The long exact sequence (5.19) now takes the form

$$\cdots \longrightarrow \mathrm{Harr}_{n+2}(C, C) \longrightarrow \mathrm{Harr}_{n+1}(C, \mathbf{k}) \longrightarrow HB_n(C) \longrightarrow \mathrm{Harr}_{n+1}(C, C) \longrightarrow \cdots$$

We refer to the original paper [37, Example 6.5] for the details of this example.

Remark 5.2.10. We defined the cyclic homology of A as the homology of the complex $CA_*(A)$. It is not difficult to see that the complex $CB_*(A)$ calculates the operadic homology [3] of the algebra A with coefficients in A.

The interpretation of $\mathcal{C}_*(A)$ is more subtle. For a $\mathcal{P}$-algebra A, the bar construction $B(A) = (B(A), d_B)$ is a differential $\mathcal{P}^!$-coalgebra (see **??** for the definitions and recall that we assume $\mathcal{P}$ to be Koszul). Our definitions can easily be dualized to introduce the cyclic homology for (differential) *co*algebras. Then $HC_*(A)$ is the cyclic cohomology of the differential $\mathcal{P}^!$-coalgebra $B(A)$. The details can be found in [37, 6.12].

Remark 5.2.11. As we already observed in **??**, we may consider a unit of an algebra either as an extra structure, or build it into the governing operad. Thus we have, for example, the operad $UAss$ for associative algebras with unit, which coincides with Ass with the exception of $UAss(0)$ which is $\mathbf{k}$ and, similarly, the operads $UComm$ and $UPoiss$. As in the case of the operadic cohomology **??**, there is a parallel cyclic homology theory for these operads. This theory agrees with the theory presented here except for the lowest term, thus we will not treat it separately here. For more details, see [37].

5.3. Modular operads.

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Let us consider again the collection $\widehat{\mathcal{M}}_0 = \{\widehat{\mathcal{M}}_0(n)\}_{n \geq 1}$ of Riemann spheres with holes. Recall **??** that $\widehat{\mathcal{M}}_0(n)$ denotes the moduli space of objects $(C; f_0, \ldots, f_n)$, where C is a complex manifold biholomorphic to $\mathbb{CP}^1$ and f_i are biholomorphic maps from the unit disk D^2 into C with disjoint images. We saw that the operation $M, N \mapsto M {}_i\circ_j N$ of gluing the

i-th hole of the surface M to the j-th hole of the surface N, defined on $\widehat{\mathcal{M}}_0$ a cyclic operad structure.

In the same manner, we may consider a single surface $M \in \widehat{\mathcal{M}}_0(n)$, choose labels i, j, $0 \leq i \neq j \leq n$, and glue the i-th hole of M along the j-th hole of the *same* surface. The result is a new surface $\xi_{ij}(M)$, with $n-2$ holes and genus 1.

This lead us to consider the collection $\widehat{\mathcal{M}} = \{\widehat{\mathcal{M}}((g,n))\}_{g\geq 0, n\geq 1}$, where $\widehat{\mathcal{M}}((g,n))$ denotes now the moduli space of genus g Riemann surfaces with $n+1$ holes. The above two families of operations act on the collection $\widehat{\mathcal{M}}$. Clearly, for $M \in \widehat{\mathcal{M}}((g,m))$ and $N \in \widehat{\mathcal{M}}(h,n)$, $M{}_i\circ_j N \in \widehat{\mathcal{M}}((g+h, m+n-1))$ and $\xi_{ij}(M) \in \widehat{\mathcal{M}}((g+1, m-2))$. Modular operads are abstractions of this structure. Our exposition follows [38]. The following definitions are made for the cateory of graded vector spaces, but they can be generalized to an arbitrary monoidal category with finite limits and colimits.

Definition 5.3.1. *A* modular collection *is a sequence* $\{\mathcal{E}((g,n))\}_{g\geq 0, n\geq 1}$ *of graded vector spaces such that each* $\mathcal{E}((g,n))$ *has a right* Σ_{n+1}*-action. Let* `MCol` *denote the category of modular collections. For* $e \in \mathcal{E}((g,n))$*, we call the number* n *the* arity *and the number* g *the* genus *of the element* e*. We also sometimes call* $n+1$ *the* valence *of* $e \in \mathcal{E}((g,n))$*.*

Remark 5.3.2. As in the cyclic case, the double bracket $((-))$ indicates the action of Σ_{n+1} instead of Σ_n. Our notation differs from that of [38] where our $\mathcal{E}((g,n))$ would be denoted $\mathcal{E}((g, n+1))$.

Thus a modular collection is a cyclic collection (Definition **??**) with an extra grading given by the genus. More precisely, each modular collection $\mathcal{E}$ determines a cyclic collection $\mathcal{E}^\flat = \{\mathcal{E}^\flat((n))\}_{n\geq 1}$ by

$$\mathcal{E}^\flat((n)) := \bigoplus_{g\geq 0} \mathcal{E}((g,n)).$$

As before, the extension functor (**??**) enables us to work with $\mathcal{E}((g,S))$ for an abitrary finite set S.

Definition 5.3.3. *A* modular operad *is a modular collection* $\mathcal{A} = \{\mathcal{A}((g,n))\}_{g\geq 0, n\geq 1}$ *together with a cyclic operad structure on the cyclic collection* $\mathcal{A}^\flat = \{\mathcal{A}^\flat((n))\}_{n\geq 1}$*, homogeneous with respect to the grading given by the genus, and a family of 'contractions'*

$$\xi_{ij} : \mathcal{A}((g,S)) \to \mathcal{A}((g+1, S - \{i,j\})),$$

defined for each finite set S *and distinct* $i, j \in S$*. The contractions are equivariant, meaning that for any* $a \in \mathcal{A}((g,S))$ *and any automorphism* σ *of* S*,*

$$\xi_{\sigma(i)\sigma(j)}(a) \cdot \sigma_{ij} = \xi_{ij}(a \cdot \sigma),$$

where σ_{ij} is the restriction $\sigma|_{S-\{i,j\}}$. We also require that, for any distinct labels $i, j, k, l \in S$, the contractions ξ_{ij} and ξ_{kl} mutually commute,

$$\xi_{ij} \circ \xi_{kl} = \xi_{kl} \circ \xi_{ij}. \tag{5.24}$$

The contractions must be compatible with the cyclic operad structure $\{{}_x\circ_y\}$ on $\mathcal{A}^\flat$ in the sense that, for any two finite sets S, T, for any $x \in S$, $y \in T$, for any distinct $i, j \in S \sqcup T - \{x, y\}$ and any $a \in \mathcal{A}((g, S))$, $b \in \mathcal{A}((h, T))$,

$$\xi_{ij}(a_x\circ_y b) = \begin{cases} (\xi_{ij}(a))_x\circ_y b, & \textit{for } i, j \in S, \\ a_x\circ_y(\xi_{ij}(b)), & \textit{for } i, j \in T, \textit{ and} \\ \xi_{xy}(a_i\circ_j b), & \textit{for } i \in S,\ j \in T. \end{cases} \tag{5.25}$$

We saw in **??** that ordinary operads were algebras over the triple of rooted trees while cyclic operads were algebras over the triple of unrooted trees **??**. We now show that also modular operads are algebras over a certain triple of graphs. The naive notion of a graph as we have used it up to this point is not subtle enough; we need to replace it by a more sophisticated notion.

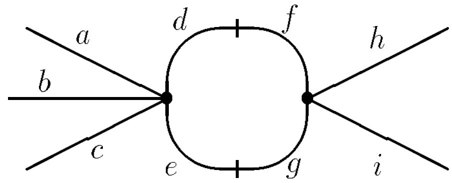
Definition 5.3.4. *A* graph Γ *is a finite set* $\mathrm{Flag}(\Gamma)$ *(whose element are called flags or half-edges) together with an involution σ and a partition λ.*

The vertices $\mathrm{Vert}(\Gamma)$ *of a graph Γ are the blocks of the partition λ. The* edges $\mathrm{Edg}(\Gamma)$ *are pairs of flags forming a two-cycle of σ relative to the decomposition of a permutation into disjoint cycles. The* legs $\mathrm{Leg}(\Gamma)$ *are the fixed-points of σ.*

We also denote by $\mathrm{Leg}(v)$ the flags belonging to the block v or, in common speech, half-edges adjacent to the vertex v. We say that two flags $x, y \in \mathrm{Flag}(\Gamma)$ *meet* if they belong to the same block of the partition λ. In plain language, this means that they share a common vertex.

We may associate to a graph Γ a finite one-dimensional cell complex $|\Gamma|$, obtained by taking one copy of $[0, \frac{1}{2}]$ for each flag and imposing the following equivalence relation: the points $0 \in [0, \frac{1}{2}]$ are identified for all flags in a block of the partition λ, and the points $\frac{1}{2} \in [0, \frac{1}{2}]$ are identified for pairs of flags exchanged by the involution σ. We call $|\Gamma|$ the *geometric realization* of the graph. We will sometimes make no distinction between the graph in the sense of Definition 5.3.4 and its geometric realization.

As an example (which we take from [38]), consider the graph with $\{a, b, \ldots, i\}$ as the set of flags, the involution $\sigma = (df)(eg)$ and the partition $\{a, b, c, d, e\} \cup \{f, g, h, i\}$. The geometric realization is then the 'sputnik'



Intuitively, a morphism $f : \Gamma_0 \to \Gamma_1$ of graphs is given by a permutation of vertices, followed by a contraction of some edges of the graph Γ_0, leaving the legs untouched. Translated into the language of Definition 5.3.4, this means an injection $f^* : \mathrm{Flag}(\Gamma_1) \to \mathrm{Flag}(\Gamma_0)$ that commutes with the involutions. Moreover, the involution σ_0 of Γ_0 must act freely on the complement of the image of f^* in (i.e. the legs of the graphs are preserved by the map f), and two flags a and b in Γ_1 meet either if $f^*(a)$ and $f^*(b)$ meet in Γ_0 or there is a chain of flags $(x_1, \ldots, x_k)$ in the complement $\mathrm{Flag}(\Gamma_0) - f^*(\mathrm{Flag}(\Gamma_1))$ of f^* in $\mathrm{Flag}(\Gamma_0)$ such that $f^*(a)$ meets x_1, $\sigma_0 x_i$ meets x_{i+1}, for $1 \leq i \leq k-1$, and $\sigma_0 x_k$ meets $f^*(b)$ (i.e. there exists a chain of edges in Γ_0 from a to b).

A morphism $f : \Gamma_0 \to \Gamma_1$ clearly defines a surjective cellular map $|f| : |\Gamma_0| \to |\Gamma_1|$ of geometric realizations, which is bijective on the legs.

There is a special class of morphisms which are given by contracting a subset of edges, without permuting the vertices. First of all, for any subset I of $\mathrm{Edg}(\Gamma)$, there is a unique graph Γ/I such that $\mathrm{Flag}(\Gamma/I)$ is obtained from $\mathrm{Flag}(\Gamma)$ by deleting the flags constituting the edges in I and combining blocks of the partition that contain flags connected by a chain in I. Then the inclusion $\mathrm{Flag}(\Gamma/I) \hookrightarrow \mathrm{Flag}(\Gamma)$ is a morphism of graphs, which we denote by $\pi_I : \Gamma \to \Gamma/I$. An important special case is when I consists of a single edge e. We then simplify our notation by writing Γ/e instead of $\Gamma/\{e\}$ and π_e instead of $\pi_{\{e\}}$.

The graph Γ/I is called the *contraction of* Γ along the set of edges I. Any morphism $f : \Gamma \to \Gamma'$ of graphs is isomorphic to a morphism of this form. This means that there exists a subset $I \subset \mathrm{Edg}(\Gamma)$ and an isomorphism $\phi : \Gamma/I \to \Gamma'$ such that the following diagram of graph maps commutes:

$$
\begin{tikzcd}
 & \Gamma/I \arrow[dr, "\phi"] & \\
\Gamma \arrow[ur, "\pi_I"] \arrow[rr, "f"'] & & \Gamma'
\end{tikzcd}
$$

Let us introduce labelled versions of the above objects. A *labelled graph* is a connected graph Γ together with a map g from $\mathrm{Vert}(\Gamma)$ the set $\{0, 1, 2, \ldots\}$. In other places of the book we consider also graphs with labelled edges, like the metrized graphs of Section **??**, but we believe that our terminology will cause no confusion.

The *genus* $g(\Gamma)$ of a labelled graph Γ is defined by the formula

$$g(\Gamma) := b_1(\Gamma) + \sum_{v \in \mathrm{Vert}(\Gamma)} g(v), \tag{5.26}$$

where $b_1(\Gamma) := \dim H_1(|\Gamma|)$ is the first Betti number of the graph $|\Gamma|$, i.e. the number of independent circuits of Γ. The classical Euler formula says that $|\mathrm{Vert}(\Gamma)| - |\mathrm{Edg}(\Gamma)| = 1 - b_1(\Gamma)$, therefore

$$b_1(\Gamma) = 1 - |\mathrm{Vert}(\Gamma)| + |\mathrm{Edg}(\Gamma)|.$$

This easily implies the following expression for $g(\Gamma)$:

$$g(\Gamma) = \sum_{v \in \mathrm{Vert}(\Gamma)} (g(v) - 1) + |\mathrm{Edg}(\Gamma)| + 1. \tag{5.27}$$

Let us derive another useful formula involving the genus. Since a flag forms either a leg or half an edge,

$$\sum_{v \in \mathrm{Vert}(\Gamma)} |\mathrm{Leg}(v)| = 2|\mathrm{Edg}(\Gamma)| + |\mathrm{Leg}(\Gamma)|.$$

Combining this with (5.27) we derive that

$$2(g(\Gamma) - 1) + |\mathrm{Leg}(\Gamma)| = \sum_{v \in \mathrm{Vert}(\Gamma)} \Big(2(g(v) - 1) + |\mathrm{Leg}(v)|\Big). \tag{5.28}$$

To modify the notion of a morphism to the category of labelled graphs, we need to observe that the preimage $f^{-1}(v)$ of a vertex $v \in \mathrm{Vert}(\Gamma_1)$ under a morphism $f : \Gamma_0 \to \Gamma_1$ is the graph consisting of those flags in Γ_0 that are connected to a flag in $\mathrm{Leg}(v)$ by a chain of edges in Γ_0 contracted by the morphism.

A *morphism* $f : \Gamma_0 \to \Gamma_1$ *of labelled graphs* is a morphism of the underlying graphs such that the genus $g(v)$ of a vertex v of Γ_1 is equal to the genus of the graph $f^{-1}(v)$, for each $v \in \mathrm{Vert}(\Gamma_1)$.

For a finite set S, let $\mathbf{\Gamma}((g, S))$ be the category whose objects are pairs $(|\Gamma|, \rho)$ consisting of a labelled graph $|\Gamma|$ of genus g and an isomorphism $\rho : \mathrm{Leg}(\Gamma) \to S$ labeling the legs of Γ by elements of S. If $S = \{0, \dots, n\}$, then we will write simply $\mathbf{\Gamma}((g, n))$ instead of $\mathbf{\Gamma}((g, S))$. Morphisms of $\mathbf{\Gamma}((g, S))$ are morphisms of labelled graphs preserving the labelling of the legs. The category $\mathbf{\Gamma}((g, S))$ has a terminal object $*_{g,S}$, the 'corolla' with no edges, one vertex v of genus g and legs labelled by S.

There is an important subcategory of $\mathbf{\Gamma}((g, S))$ consisting of *stable* graphs. A graph Γ is called stable if

$$2(g(v) - 1) + |\mathrm{Leg}(v)| > 0, \text{ at each vertex } v \in \mathrm{Vert}(\Gamma).$$

The subcategory of stable graphs has the property that, for each fixed $g \geq 0$ and a finite set S, there is only finite number of isomorphism classes of stable graphs $\Gamma \in \mathbf{\Gamma}((g, S))$ [38, Lemma 2.16].

The stability property of graphs is an analog of the stability of Riemann surfaces. (Recall that a genus g Riemann surface with k punctures is called stable if $2(g-1)+k > 0$.) In fact, all constructions in [38] are made for the subcategory of stable graphs, since the authors are primarily concerned with applications in algebraic geometry.

For a modular collection $\mathcal{E} = \{\mathcal{E}((g,n))\}_{g\geq 0, n\geq 1}$ and a labelled graph Γ, let $\mathcal{E}((\Gamma))$ be the tensor product

$$\mathcal{E}((\Gamma)) := \bigotimes_{v\in \mathrm{Vert}(\Gamma)} \mathcal{E}((g(v), \mathrm{Leg}(v))). \tag{5.29}$$

For a category $\mathcal{C}$, let $\mathrm{Iso}(\mathcal{C})$ denote the subcategory all of whose morphisms are isomorphisms. Evidently, the correspondence $\Gamma \mapsto \mathcal{E}((\Gamma))$ defines a functor from the category $\mathrm{Iso}(\mathbf{\Gamma}((g,S)))$ to the category of vector spaces and their isomorphisms. We may thus define an endofunctor $\mathbb{M}$ on the category `MCol` of modular collections by the formula

$$\mathbb{M}\mathcal{E}((g,S)) = \operatorname*{colim}_{\Gamma\,\in\,\mathrm{Iso}\mathbf{\Gamma}((g,S))} \mathcal{E}((\Gamma)). \tag{5.30}$$

This, by definition, means that $\mathbb{M}\mathcal{E}((g,S))$ is the quotient of the space $\bigoplus_{\Gamma\in\mathbf{\Gamma}((g,S))}\mathcal{E}((g,S))$ modulo the identifications $\mathcal{E}((\Gamma_0)) \ni x \sim \mathcal{E}((f))(x) \in \mathcal{E}((\Gamma_1))$, for any isomorphism $f : \Gamma_0 \to \Gamma_1$. Choosing a representative Γ_γ for any isomorphism class γ in $\mathbf{\Gamma}((g,S))$, we obtain a noncanonical identification

$$\mathbb{M}\mathcal{E}((g,S)) \cong \bigoplus_{\gamma\in\{\mathbf{\Gamma}((g,S))\}} \mathcal{E}((\Gamma_\gamma))_{\mathrm{Aut}(\Gamma_\gamma)}, \tag{5.31}$$

where $\{\mathbf{\Gamma}((g,S))\}$ denotes the set of isomorphism classes of objects of the category $\mathbf{\Gamma}((g,S))$ and the subscript $\mathrm{Aut}(\Gamma_\gamma)$ denotes the space of coinvariants.

We will show that the functor $\mathbb{M}$ is a triple in the sense of Definition **??**. Our arguments here, as everywhere in this section, closely follow [38].

Recall that the *nerve* of a category $\mathcal{C}$ is a simplicial category $\mathrm{Nerve}_*(\mathcal{C})$ whose category of k-simplices, $\mathrm{Nerve}_k(\mathcal{C})$, is the category of diagrams

$$(f_1,\ldots,f_k) = [\Gamma_0 \xrightarrow{f_1} \Gamma_1 \xrightarrow{f_2} \ldots \xrightarrow{f_{k-1}} \Gamma_{k-1} \xrightarrow{f_k} \Gamma_k]$$

in $\mathcal{C}$, while the morphisms are morphisms of such diagrams. The face functors

$$\partial_i : \mathrm{Nerve}_k(\mathcal{C}) \to \mathrm{Nerve}_{k-1}(\mathcal{C})$$

are given, for $0 \leq i \leq k$, by the usual formulas

$$\partial_i(f_1,\ldots,f_k) =: \begin{cases} (f_2,\ldots,f_k), & i = 0, \\ (f_1,\ldots,f_{i+1}f_i,\ldots,f_k), & 1 \leq i \leq k-1, \\ (f_1,\ldots,f_{k-1}), & i = k. \end{cases} \tag{5.32}$$

For $k = 1$ we interpret the above formulas as $\partial_0(\Gamma_0 \xrightarrow{f} \Gamma_1) := \Gamma_1$ and $\partial_1(\Gamma_0 \xrightarrow{f} \Gamma_1) := \Gamma_0$. Similarly, the degeneracy functors $\sigma_j : \mathrm{Nerve}_k(\mathcal{C}) \to \mathrm{Nerve}_{k+1}(\mathcal{C})$ are, for $0 \leq j \leq k$, given by

$$\sigma_j(f_1,\ldots,f_k) := (f_1,\ldots,f_j,\mathbb{1}_{\Gamma_j},f_{j+1},\ldots,f_k).$$

Finally, let $\mathrm{Iso}_*(\mathcal{C})$ denote the simplicial category $\mathrm{Iso}(\mathrm{Nerve}_*(\mathcal{C}))$. The following proposition is due to Getzler and Kapranov [38].

Proposition 5.3.5. *For each $k \geq 0$, the $(k+1)$-th iterate of the functor $\mathbb{M}$ is given by*

$$(\mathbb{M}^{k+1}\mathcal{E})((g,S)) = \operatorname*{colim}_{[\Gamma_0 \xrightarrow{f_1} \dots \xrightarrow{f_k} \Gamma_k] \in \mathrm{Iso}_k \mathbf{\Gamma}((g,S))} \mathcal{E}((\Gamma_0)). \tag{5.33}$$

Proof. We prove the proposition by induction. For $k=0$, it is the definition. Fix an $m \geq 1$ and suppose we have proved (5.33) for $k < m$. Let us prove it for $k = m$.

For $[\Gamma_0 \xrightarrow{f_1} \Gamma_1 \xrightarrow{f_2} \dots \xrightarrow{f_{m-1}} \Gamma_{m-1} \xrightarrow{f_m} \Gamma_m] \in \mathrm{Iso}_m(\mathbf{\Gamma}((g,S)))$ and $u \in \mathrm{Vert}(\Gamma_m)$, let

$$[\Gamma^u_0 \xrightarrow{f^u_1} \Gamma^u_1 \xrightarrow{f^u_2} \dots \xrightarrow{f^u_{m-1}} \Gamma^u_{m-1}] \in \mathrm{Iso}_{m-1}(\mathbf{\Gamma}((g(u), \mathrm{Leg}(u)))) \tag{5.34}$$

be defined by $\Gamma^u_i =: (f_m \cdots f_{i+1})^{-1}(u)$, while the maps $f^u_i : \Gamma^u_{i-1} \to \Gamma^u_i$ are the restrictions, $0 \leq i \leq m-1$. Then

$$\begin{aligned}
\operatorname*{colim}_{[\Gamma_0 \xrightarrow{f_1} \dots \xrightarrow{f_m} \Gamma_m] \in \mathrm{Iso}_m \mathbf{\Gamma}((g,S))} \mathcal{E}((\Gamma_0)) &= \operatorname*{colim}_{[\Gamma_0 \xrightarrow{f_1} \dots \xrightarrow{f_m} \Gamma_m] \in \mathrm{Iso}_m \mathbf{\Gamma}((g,S))} \left\{ \bigotimes_{v \in \mathrm{Vert}(\Gamma_0)} \mathcal{E}((g(v), \mathrm{Leg}(v))) \right\} \\
&= \operatorname*{colim}_{[\Gamma_0 \xrightarrow{f_1} \dots \xrightarrow{f_m} \Gamma_m] \in \mathrm{Iso}_m \mathbf{\Gamma}((g,S))} \left\{ \bigotimes_{u \in \mathrm{Vert}(\Gamma_m)} \bigotimes_{v \in \mathrm{Vert}(\Gamma^u_0)} \mathcal{E}((g(v), \mathrm{Leg}(v))) \right\} \\
&= \operatorname*{colim}_{[\Gamma_0 \xrightarrow{f_1} \dots \xrightarrow{f_m} \Gamma_m] \in \mathrm{Iso}_m \mathbf{\Gamma}((g,S))} \left\{ \bigotimes_{u \in \mathrm{Vert}(\Gamma_m)} \mathcal{E}((\Gamma^u_0)) \right\}.
\end{aligned} \tag{5.35}$$

Observe now that the correspondence

$$[\Gamma_0 \xrightarrow{f_1} \Gamma_1 \xrightarrow{f_2} \dots \xrightarrow{f_{m-1}} \Gamma_{m-1} \xrightarrow{f_m} \Gamma_m] \longmapsto \Gamma_m \times \bigtimes_{u \in \mathrm{Vert}(\Gamma_m)} [\Gamma^u_0 \xrightarrow{f^u_1} \Gamma^u_1 \xrightarrow{f^u_2} \dots \xrightarrow{f^u_{m-1}} \Gamma^u_{m-1}]$$

defines an isomorphism of the categories,

$$\mathrm{Iso}_m \mathbf{\Gamma}((g,S)) \cong \mathrm{Iso}_m \mathbf{\Gamma}((g,S)) \times \bigtimes_{u \in \mathrm{Vert}(\Gamma_n)} \mathrm{Iso}_{m-1} \mathbf{\Gamma}((g(u), \mathrm{Leg}(u))).$$

So, we can rewrite the last expression of (5.35) as

$$\begin{aligned}
&\operatorname*{colim}_{\Gamma_m \in \mathrm{Iso}(\mathbf{\Gamma}((g,S)))} \left\{ \bigotimes_{u \in \mathrm{Vert}(\Gamma_m)} \left[\operatorname*{colim}_{[\Gamma^u_0 \xrightarrow{f^u_1} \dots \xrightarrow{f^u_{m-1}} \Gamma^u_{m-1}] \in \mathrm{Iso}_{k-1} \Gamma((g(u), \mathrm{Leg}(u)))} \mathcal{E}((\Gamma^u_0)) \right] \right\} = \\
&= \operatorname*{colim}_{\Gamma_m \in \mathrm{Iso}(\mathbf{\Gamma}((g,S)))} \left\{ \bigotimes_{u \in \mathrm{Vert}(\Gamma_m)} \mathbb{M}^m \mathcal{E}((g(v)), \mathrm{Leg}(u))) \right\} = \mathbb{M}^{m+1}\mathcal{E}((g,S)). \qquad \blacksquare
\end{aligned}$$

It might be helpful to compare the construction of Proposition 5.3.5 with the iteration of the tensor algebra triple for associative **k**-algebras, where there is a 'nesting' of tensor algebras: tensor algebra of tensor algebra of tensor algebra ...

An m-simplex $[\Gamma_0 \xrightarrow{f_1} \Gamma_1 \xrightarrow{f_2} \ldots \xrightarrow{f_{m-1}} \Gamma_{m-1} \xrightarrow{f_m} \Gamma_m]$ defines a similar nesting of tensor products given by the nested chains of subsets of the vertices of Γ_0:

$$(f_m \cdots f_1)^{-1}(u_m) \supset (f_{m-1} \cdots f_1)^{-1}(u_{m-1}) \supset \cdots f_1^{-1}(u_1) \supset \{u_0\},$$

where $u_m \in \mathrm{Vert}(\Gamma_m)$, $u_{m-1} \in f_m^{-1}(u_m)$, ..., $u_1 \in f_2^{-1}(u_2)$ and $u_0 \in f_1^{-1}(u_1)$.

Let us recall **??** that a triple in a category $\mathcal{C}$ is an is an endofunctor T on $\mathcal{C}$ together with natural transformations $\mu : T^2 \to T$ and $v : 1\!\!1_{\mathcal{C}} \to T$ such that (T, μ, v) is an associative monoid in the category of endofunctors of $\mathcal{C}$.

Define $\mu : \mathbb{M}^2\mathcal{E} \to \mathbb{M}\mathcal{E}$ to be the transformation induced by $\partial_1 : \mathrm{Iso}_1\mathbf{\Gamma}((g,S)) \to \mathrm{Iso}_0\mathbf{\Gamma}((g,S))$, which sends $[\Gamma_0 \xrightarrow{f_1} \Gamma_1]$ to Γ_0. The unit $v : \mathcal{E} \to \mathbb{M}\mathcal{E}$ is defined as the inclusion of the summand $\mathcal{E}((*_{g,S})) \cong \mathcal{E}((g,S))$ of $\mathbb{M}\mathcal{E}((g,S))$ associated to the corolla $*_{g,S}$.

Theorem 5.3.6. *The transformations $\mu : \mathbb{M}^2 \to \mathbb{M}$ and $v : 1\!\!1_{\mathtt{MCol}} \to \mathbb{M}$ defined above form a triple $(\mathbb{M}, \mu, v)$ in the category of modular collections.*

Proof. The natural transformations $\mathbb{M}\mu$ and $\mu\mathbb{M} : (\mathbb{M}^3\mathcal{E})((g,S)) \to (\mathbb{M}^2\mathcal{E})((g,S))$ are clearly induced by the functors $\partial_1, \partial_2 : \mathrm{Iso}_2\mathbf{\Gamma}((g,S)) \to \mathrm{Iso}_1\mathbf{\Gamma}((g,S))$, which send the sequence $\Gamma_0 \xrightarrow{f_1} \Gamma_1 \xrightarrow{f_2} \Gamma_2$ respectively to $\Gamma_0 \xrightarrow{f_2 f_1} \Gamma_2$ and $\Gamma_0 \xrightarrow{f_1} \Gamma_1$. In the same manner, the natural transformations $\mu(\mu\mathbb{M})$ and $\mu(\mathbb{M}\mu)$ are induced respectively by $\partial_1\partial_2$ and $\partial_1\partial_1$. Since $\partial_1\partial_2 = \partial_1\partial_1$ as it must be in any simplicial category (in fact, both compositions send $\Gamma_0 \xrightarrow{f_1} \Gamma_1 \xrightarrow{f_2} \Gamma_2$ to Γ_0), $\mu(\mu\mathbb{M}) = \mu(\mathbb{M}\mu)$ which proves the associativity of μ. To see that v is a unit, i.e. that $\mu(v\mathbb{M}) = \mu(\mathbb{M}v) = \mathbb{M}$, is immediate. ∎

An algebra over the triple $(\mathbb{M}, \mu, v)$ is, by definition (see **??**), a modular collection $\mathcal{A} = \{\mathcal{A}((g,n))\}_{g \geq 0, n \geq 1}$ together with a morphism of collections $\alpha : \mathbb{M}(\mathcal{A}) \to \mathcal{A}$ satisfying

$$\alpha \circ \mathbb{M}(\alpha) = \alpha \circ \mu_{\mathcal{A}}, \text{ and} \tag{5.36}$$

$$\alpha v_{\mathcal{A}} = 1\!\!1_{\mathcal{A}}. \tag{5.37}$$

The following theorem is a modular version of Theorem **??** resp. Theorem **??**.

Theorem 5.3.7. *Modular operads are algebras over the triple $(\mathbb{M}, \mu, v)$ in the category of modular collections.*

Before proving the theorem, we formulate a technical proposition. Suppose we have a map $\alpha : \mathbb{M}\mathcal{A} \to \mathcal{A}$ of modular collections satisfying (5.37), but not necessarily (5.36). Each such a map defines, for a graph $\Gamma \in \mathbf{\Gamma}((g,S))$, a map $\alpha_\Gamma : \mathcal{A}((\Gamma)) \to \mathcal{A}((g,S))$, as the composition of the universal map $\mathcal{A}((\Gamma)) \to \mathbb{M}\mathcal{A}((g,S))$ with α. We call this map the *composition along the graph* Γ. Observe that (5.37) requires that the composition along the corolla be given by the identification $\mathcal{A}((*_{g,S})) \cong \mathcal{A}((g,S))$.

We may define, for each morphism $f : \Gamma_0 \to \Gamma_1$, a morphism $\alpha((f)) : \mathcal{A}((\Gamma_0)) \to \mathcal{A}((\Gamma_1))$ as the composition

$$\begin{aligned}\mathcal{A}((\Gamma_0)) &= \bigotimes_{u \in \mathrm{Vert}(\Gamma_0)} \mathcal{A}((g(u), \mathrm{Leg}(u))) \cong \bigotimes_{v \in \mathrm{Vert}(\Gamma_1)} \mathcal{A}((f^{-1}(v))) \\ &\xrightarrow{\bigotimes_v \alpha_{f^{-1}(v)}} \bigotimes_{v \in \mathrm{Vert}(\Gamma_1)} \mathcal{A}((g(v), \mathrm{Leg}(v))) = \mathcal{A}((\Gamma_1)).\end{aligned} \tag{5.38}$$

The following statement is a 'modular' version of Proposition **??**.

Proposition 5.3.8. *A map $\alpha : \mathbb{M}\mathcal{A} \to \mathcal{A}$ of modular collections satisfies (5.36), i.e. it is an algebra over the triple $(\mathbb{M}, \mu, v)$, if and only if the correspondence $\Gamma \mapsto \mathcal{A}((\Gamma))$, $f \mapsto \alpha(f)$ defines, for each $g \geq 0$ and a finite set S, a functor from the category $\mathbf{\Gamma}((g, S))$ to the category of graded vector spaces, that is, if*

$$\alpha((f_2 f_1)) = \alpha((f_2))\alpha((f_1)) \tag{5.39}$$

for any diagram $\Gamma_0 \xrightarrow{f_1} \Gamma_1 \xrightarrow{f_2} \Gamma_2$ in $\mathbf{\Gamma}((g, S))$.

Proof. Consider a diagram $\Gamma_0 \xrightarrow{f_1} \Gamma_1 \xrightarrow{f_2} \Gamma_2$. By definition, $\alpha((f_2))\alpha((f_1))$ is the composition

$$\begin{aligned}\mathcal{A}((\Gamma_0)) &= \bigotimes_{w \in \mathrm{Vert}(\Gamma_0)} \mathcal{A}((g(w), \mathrm{Leg}(w))) \cong \bigotimes_{v \in \mathrm{Vert}(\Gamma_1)} \mathcal{A}((f_1^{-1}(v))) \\ &\xrightarrow{\bigotimes_v \alpha_{f_1^{-1}(v)}} \bigotimes_{v \in \mathrm{Vert}(\Gamma_1)} \mathcal{A}((g(v), \mathrm{Leg}(v))) \cong \bigotimes_{u \in \mathrm{Vert}(\Gamma_2)} \mathcal{A}((f_2^{-1}(u))) \\ &\xrightarrow{\bigotimes_u \alpha_{f_2^{-1}(u)}} \bigotimes_{u \in \mathrm{Vert}(\Gamma_2)} \mathcal{A}((g(u), \mathrm{Leg}(u))) = \mathcal{A}((\Gamma_2)).\end{aligned} \tag{5.40}$$

Our next task is to interpret $\alpha((f_2))\alpha((f_1))$ in terms of $\alpha \circ \mathbb{M}$. For $u \in \mathrm{Vert}(\Gamma_2)$, let $[\Gamma_0^u \xrightarrow{f_1^u} \Gamma_1^u] \in \mathrm{Iso}_1\mathbf{\Gamma}((g(u), \mathrm{Leg} u))$ be as in (5.34), i.e. $\Gamma_1^u := f_2^{-1}(u)$, $\Gamma_0^u := (f_2 f_1)^{-1}(u)$ and $f_1^u =$ the restriction of f_1 to Γ_0^u. Recall from Proposition (5.3.5) that, for each $u \in \mathrm{Vert}(\Gamma_2)$,

$$(\mathbb{M}^2\mathcal{A})((g(u), \mathrm{Leg}(u))) = \operatorname*{colim}_{[\Upsilon_0 \xrightarrow{g} \Upsilon_1] \in \mathrm{Iso}_1\mathbf{\Gamma}((g(u), \mathrm{Leg}(u)))} \mathcal{A}((\Upsilon_0)).$$

This means that there is a map $\rho_u : \mathcal{A}((\Gamma_0^u) \to (\mathbb{M}^2\mathcal{A})((g(u), \mathrm{Leg}(u)))$ given by mapping $\mathcal{A}((\Gamma_0^u)$ to the component indexed by $[\Gamma_0^u \xrightarrow{f_1^u} \Gamma_1^u] \in \mathrm{Iso}_1\mathbf{\Gamma}((g(u), \mathrm{Leg}(u)))$. Let us define $[\alpha \circ \mathbb{M}(\alpha)]_{f_1^u} : \mathcal{A}((\Gamma_0^u)) \to \mathcal{A}((g(u), \mathrm{Leg}(u)))$ as the composition

$$\mathcal{A}((\Gamma_0^u)) \xrightarrow{\rho_u} (\mathbb{M}^2\mathcal{A})((g(u), \mathrm{Leg}(u))) \xrightarrow{\alpha \circ \mathbb{M}(\alpha)} \mathcal{A}((g(u), \mathrm{Leg}(u))$$

A moment's reflection on (5.40) persuades one that

$$\alpha((f_2))\alpha((f_1)) = \bigotimes_{u\in \mathrm{Vert}(\Gamma_0)} [\alpha \circ \mathbb{M}(\alpha)]_{f_1^u}. \tag{5.41}$$

Similarly, $\alpha((f_2 f_1))$ is, by definition, given by

$$\begin{aligned}\mathcal{A}((\Gamma_0)) &= \bigotimes_{w\in \mathrm{Vert}(\Gamma_0)} \mathcal{A}((g(w), \mathrm{Leg}(w))) \cong \bigotimes_{u\in \mathrm{Vert}(\Gamma_2)} \mathcal{A}(((f_2 f_1)^{-1}(u))) \\ &\xrightarrow{\bigotimes_u \alpha_{(f_2 f_1)^{-1}(u)}} \bigotimes_{u\in \mathrm{Vert}(\Gamma_2)} \mathcal{A}((g(u), \mathrm{Leg}(u))) = \mathcal{A}((\Gamma_2)).\end{aligned}$$

Similarly as above we see that if we denote by $[\alpha \circ \mu_{\mathcal{A}}]_{f_1^u}$ the composition

$$\mathcal{A}((\Gamma_0^u)) \xrightarrow{\rho_u} (\mathbb{M}^2\mathcal{A})((g(u), \mathrm{Leg}(u))) \xrightarrow{\alpha\circ\mu_{\mathcal{A}}} \mathcal{A}((g(u), \mathrm{Leg}(u))),$$

then

$$\alpha((f_2 f_1)) = \bigotimes_{u\in \mathrm{Vert}(\Gamma_2)} [\alpha \circ \mu_{\mathcal{A}}]_{f_1^u}. \tag{5.42}$$

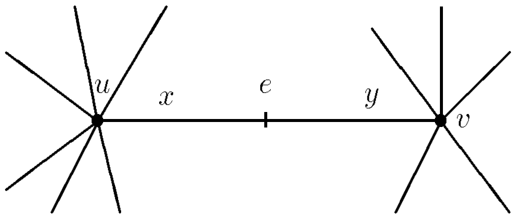
This together with (5.41) implies that if $\alpha \circ \mathbb{M}(\alpha) = \alpha \circ \mu_{\mathcal{A}}$, then $\alpha((f_2 f_1)) = \alpha((f_2))\alpha((f_1))$, that is $\alpha((-))$ is functorial.

Suppose, on the other hand, that $\alpha((-))$ is functorial. Let $[\Gamma_0 \xrightarrow{f} \Gamma_1] \in \mathrm{Iso}_1\mathbf{\Gamma}((g,S))$ and apply the above considerations to $[\Gamma_0 \xrightarrow{f_1} \Gamma_1 \xrightarrow{f_2} \Gamma_2]$ with $f_1 := f$, $\Gamma_2 := *_{g,S}$ and $f_2 : \Gamma_1 \to \Gamma_2$ the contraction. For the unique vertex $u \in \mathrm{Vert}(*_{g,S})$ we of course have $[\Gamma_0^u \xrightarrow{f_1^u} \Gamma_1^u] = [\Gamma_0 \xrightarrow{f} \Gamma_1]$. Equations (5.41) and (5.42) imply that the following diagram is commutative:

$$\begin{tikzcd}
(\mathbb{M}^2\mathcal{A})((g,S)) \arrow[d, "\alpha\circ\mathbb{M}(\alpha)"'] & \mathcal{A}((\Gamma_0)) \arrow[l] \arrow[r] \arrow[d, "\alpha((f_2f_1))=\alpha((f_2))\alpha((f_1))"] & (\mathbb{M}^2\mathcal{A})((g,S)) \arrow[d, "\alpha\circ\mu_{\mathcal{A}}"] \\
\mathcal{A}((g,S)) \arrow[r, "\cong"] & \mathcal{A}((*_{g,S})) & \mathcal{A}((g,S)) \arrow[l, "\cong"']
\end{tikzcd}$$

To finish the proof, observe that $(\mathbb{M}^2\mathcal{A})((g,S))$ is the colimit of $\mathcal{A}((\Gamma_0))$ over all such $[\Gamma_0 \xrightarrow{f} \Gamma_1] \in \mathrm{Iso}_1\mathbf{\Gamma}((g,S))$. ■

Proof of Theorem 5.3.7. Suppose that $\alpha : \mathbb{M}\mathcal{A} \to \mathcal{A}$ is an algebra over the triple $(\mathbb{M}, \mu, v)$. By Proposition 5.3.8, α then defines a functor from the category $\mathbf{\Gamma}((g,S))$ to `MCol`. Let us explain how α induces the data of Definition 5.3.3 by evaluating α on $\mathcal{A}((\Upsilon))$ for graphs Υ with one edge e. The composition ${}_x\circ_y$ is defined using Υ of the form:

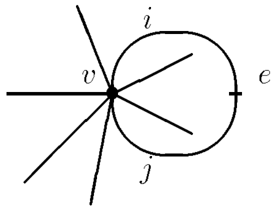


(5.43)

with two distinct vertices u and v and flags $x \in \mathrm{Leg}(u)$ and $y \in \mathrm{Leg}(v)$ forming the edge e, as the composition along Υ:

$$ {}_x\circ_y := \mathcal{A}((g_1, \mathrm{Vert}(u))) \otimes \mathcal{A}((g_2, \mathrm{Vert}(v))) \cong \mathcal{A}((\Upsilon)) \xrightarrow{\alpha_\Upsilon} \mathcal{A}((g_1 + g_2, \mathrm{Leg}(u) \sqcup \mathrm{Leg}(v) - \{x, y\})), $$

where $g_1 := g(u)$ and $g_2 := (v)$. Similarly, the contractions ξ_{ij} are defined as the composition along the graph Θ:



(5.44)

with one vertex v of genus g and flags $i, j \in \mathrm{Leg}(v)$ forming the edge e, by

$$ \xi_{ij} := \mathcal{A}((g, \mathrm{Vert}(v))) \cong \mathcal{A}((\Theta)) \xrightarrow{\alpha_\Theta} \mathcal{A}(g + 1, \mathrm{Vert}(v) - \{i, j\}). $$

We must prove that the operations defined above satisfy the axioms of Definition 5.3.3, i.e. that the compositions $\{{}_x\circ_y\}$ form an operad structure graded by the genus and that the contractions $\{\xi_{ij}\}$ satisfy (5.24) and (5.25).

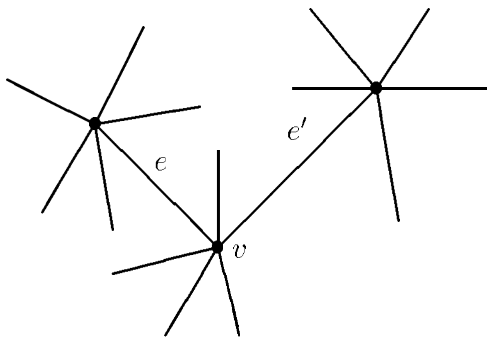
Let $\Gamma \in \mathbf{\Gamma}((g, S))$ be a graph with two distinct edges $e, e' \in \mathrm{edg}(\Gamma)$. The projections π_e and $\pi_{e'}$ obviously commute:

$$ \begin{tikzcd} \Gamma \arrow[r, "\pi_{e'}"] \arrow[d, "\pi_e"'] & \Gamma/e' \arrow[d, "\pi_e"] \\ \Gamma/e \arrow[r, "\pi_{e'}"] & \Gamma/\{e, e'\} \end{tikzcd} $$

where we of course identify $(\Gamma/e)/e' \cong (\Gamma/e')/e \cong \Gamma/\{e, e'\}$. Since α is a functor, the commutativity of the above diagram implies that

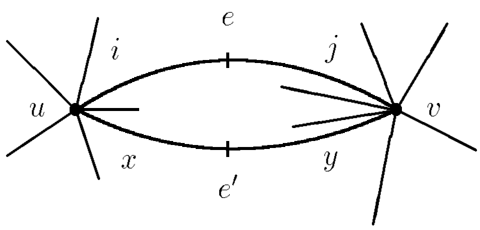
$$ \alpha((\pi_e))\alpha((\pi_{e'})) = \alpha((\pi_{e'}))\alpha((\pi_e)). \tag{5.45} $$

Evaluating (5.45) on graphs such as



(5.46)

where both e and e' have two distinct vertices and they meet in one vertex v, we prove that $\mathcal{A}^\flat$ with operations $\{{}_x\circ_y\}$ form a cyclic operad graded by the genus. The argument is exactly the same as in **??**, the only difference is that we must also check that the operations ${}_x\circ_y$ are homogeneous with respect to the genus, but this is obvious. Consider a graph Γ as in the following picture:

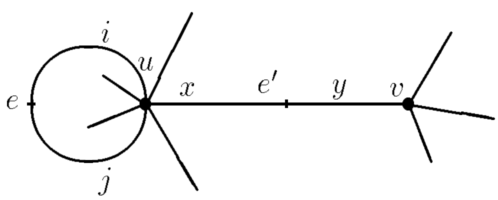


(5.47)

The evaluation of the left hand side of equation (5.45) on

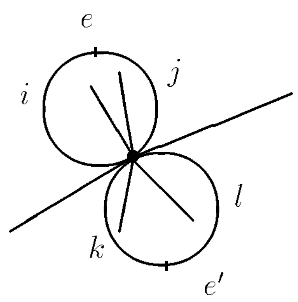
$$a \otimes b \in \mathcal{A}((g, \mathrm{Leg}(u))) \otimes \mathcal{A}((h, \mathrm{Leg}(v))) \cong \mathcal{A}((\Gamma))$$

for Γ gives $\xi_{ij}(a_x \circ_y b)$, while the evaluation of the right hand side gives $\xi_{xy}(a_i \circ_j b)$, and we obtain the third case of (5.25). The evaluation of the left hand side of (5.45) on the graph



(5.48)

gives $\xi_{ij}(a_x \circ_y b)$, while the right hand side gives $(\xi_{ij}(a))_x \circ_y b$, which is the first case of (5.25). The evaluation of (5.45) on the same graph but with the indices u and v, x and y interchanged gives the second case of (5.25). Finally, the evaluation of (5.45) on



(5.49)

gives (5.24).

On the other hand, suppose that $\mathcal{A}^\flat$ is a graded (with respect to the genus) cyclic operad equipped with contraction maps $\{\xi_{ij}\}$ that satisfy (5.24) and (5.25). We will show how, for each $g \geq 0$ and each finite set S, the modular operad structure defines a functor from the category $\mathbf{\Gamma}(\!(g,S)\!)$ to $\mathtt{MCol}$. On objects, we define the functor to be $\mathcal{A}(\!(\Gamma)\!)$ as in (5.29). This definition is clearly functorial on isomorphisms $\phi : \Gamma_0 \to \Gamma_1$.

As the next step we define $\mathcal{A}(\!(f)\!)$ for f a contraction along an edge e of a graph Γ, $f = \pi_e : \Gamma \to \Gamma/e$. Let Υ be the minimal subgraph of Γ containing e and all half-edges (flags) meeting e.

If the edge e has two distinct ends u and v, then Υ looks as in (5.43). We define $\alpha(\!(f)\!)$ to be the operadic composition ${}_x\circ_y$ on Υ and the identity outside Υ. Formally this means that we realize that

$$\begin{aligned} \mathcal{A}(\!(\Gamma)\!) &\cong \mathcal{A}(\!(g_1,u)\!)\otimes\mathcal{A}(\!(g_2,v)\!)\otimes \bigotimes_{w\neq u,v} \mathcal{A}(\!(g(w),\mathrm{Leg}(w))\!), \text{ and} \\ \mathcal{A}(\!(\Gamma/e)\!) &\cong \mathcal{A}(\!(g_1+g_2,\mathrm{Leg}(\bar v))\!)\otimes \bigotimes_{w\neq u,v} \mathcal{A}(\!(g(w),\mathrm{Leg}(w))\!), \end{aligned}$$

where $\overline{v}$ is the vertex of Γ/e created by the contraction of e. Then we define

$$\mathcal{A}(\!(\pi_e)\!) := {}_x\circ_y \otimes \bigotimes_{w\neq u,v} 1\!\!1_{\mathcal{A}(\!(g(w),\mathrm{Leg}(w))\!)}. \tag{5.50}$$

Another possibility is that the edge e forms a loop with one vertex v. The graph Υ then looks as in (5.44). Exactly as in (5.50), we define $\mathcal{A}(\!(f)\!)$ to be the contraction ξ_{ij} on Υ and the identity outside Υ.

A general morphism $f : \Gamma_0 \to \Gamma_1$ of graphs decomposes into a composition

$$\Gamma_0 \xrightarrow{\pi_I} \Gamma_0/I \xrightarrow{\phi} \Gamma_1,$$

where I is the set of edges contracted by f and $\phi : \Gamma_0/I \to \Gamma_1$ is an isomorphism. Choosing an ordering $\{e_1, \ldots, e_k\}$ of the edges in I, we obtain a factorization

$$\Gamma_0 \xrightarrow{\pi_{e_1}} \Gamma_0/e_1 \xrightarrow{\pi_{e_2}} \Gamma_0/\{e_1,e_2\} \xrightarrow{\pi_{e_3}} \ldots \xrightarrow{\pi_{e_k}} \Gamma_0/I \xrightarrow{\phi} \Gamma_1,$$

where each morphism is a contraction along one edge except the last, which is an isomorphism. We define

$$\alpha((f)) := \mathcal{A}((\phi))\mathcal{A}((\pi_{e_k}))\ldots\mathcal{A}((\pi_{e_1})).$$

We must prove that this definition is independent of the ordering of the elements of I. Of course, it suffices to prove the composition does not change if we interchange two consecutive edges e_i and e_{i+1}. This is evident if e_i and e_{i+1} have no common vertex. If they have, then they form a subgraph of Γ with two edges, which must one of the graphs listed in (5.47)–(5.49). But then the maps $\mathcal{A}((\pi_{e_i}))$ and $\mathcal{A}((\pi_{e_{i+1}}))$ commute, because the axioms of modular operads were derived exactly as those assuring the commutativity over these graphs.

Thus, we have defined a functor $\Gamma \mapsto \mathcal{A}((\Gamma))$, $f \mapsto \alpha((f))$ on the category $\mathbf{\Gamma}((g,S))$, using the operad structure of $\mathcal{A}^\flat$ and the contractions ξ_{ij}. By Proposition 5.3.8, this functor is an algebra over the triple $(\mathbb{M},\mu,v)$. ■

Example 5.3.9. Let $V=(V,B)$ be a differential graded vector space with a graded symmetric inner product $B: V\otimes V \to \mathbf{k}$. Recall that graded symmetry means that, for any homogeneous $u,v\in V$,

$$B(u,v) = (-1)^{|u||v|}B(v,u).$$

Let us define, for each genus $g\geq 0$ and a finite set S,

$$End_V((g,S)) := V^{\otimes S}.$$

It follows from definition that, for any graph $\Gamma\in\mathbf{\Gamma}((g,S))$, $End_V((\Gamma)) = V^{\otimes \mathrm{Flag}(\Gamma)}$.

Let $B^{\otimes \mathrm{Edg}(\Gamma)} : V^{\otimes \mathrm{Flag}(\Gamma)} \to V^{\otimes \mathrm{Leg}(\Gamma)}$ be the multilinear form which contracts the factors of $V^{\otimes \mathrm{Flag}(\Gamma)}$ corresponding to the flags which are paired up as edges of Γ. Then we define $\alpha_\Gamma : End_V((g,\Gamma)) \to End_V((g,S))$ to be the map

$$\alpha_\Gamma : End_V((\Gamma)) \cong V^{\otimes \mathrm{Flag}(\Gamma)} \xrightarrow{B^{\otimes \mathrm{Edg}(\Gamma)}} V^{\otimes \mathrm{Leg}(\Gamma)} \cong V^{\otimes S} = End_V((g,S)).$$

It is easy to show that the compositions $\{\alpha_\Gamma|\ \Gamma\in\mathbf{\Gamma}((g,S))\}$ define on End_V the structure of a modular operad. The underlying cyclic operad **??** coincides with the cyclic endomorphism operad introduced in **??**.

Definition 5.3.10. *A modular algebra over a modular operad $\mathcal{A}$ is given by a graded vector space with a symmetric inner product, $V=(V,B)$, and a map $a:\mathcal{A}\to End_V$ of modular operads.*

In the rest of this section we introduce the Feynman transform of a modular operad. It is an analog of the bar construction of an ordinary operad **??**. The Feynman transform was introduced by E. Getzler and M.M. Kapranov in [38] and called so because it is given as a sum over graphs, as is Feynman's expansion for amplitudes in quantum field theory.

The conceptual difference between the operadic bar construction and the Feynman transform $\mathsf{F}\mathcal{A}$ of a modular operad is that $\mathsf{F}\mathcal{A}$ is not a modular operad, but a certain type of a 'twisted' modular operad. To give a precise meaning to this, we need to introduce some technical notions. Suppose we are given, for each $g \geq 0$, $n \geq 1$, a functor

$$\mathfrak{D} : \mathrm{Iso}\,\mathbf{\Gamma}(\!(g,n)\!) \to \mathtt{gVect}$$

from $\mathrm{Iso}\,\mathbf{\Gamma}(\!(g,n)\!)$ to the category $\mathtt{gVect}$ of graded vector spaces. For each $k \geq 0$, there is a natural extension of $\mathfrak{D}$ to a functor $\mathfrak{D}_k : \mathrm{Iso}_k\mathbf{\Gamma}(\!(g,n)\!) \to \mathtt{gVect}$ given by the formula

$$\mathfrak{D}_k([\Gamma_0 \xrightarrow{f_1} \Gamma_1 \xrightarrow{f_2} \cdots \xrightarrow{f_k} \Gamma_k]) :=$$
$$:= \mathfrak{D}(\Gamma_k) \otimes \bigotimes_{v_k \in \mathrm{Vert}(\Gamma_k)} \mathfrak{D}(f_k^{-1}(v_k)) \otimes \cdots \otimes \bigotimes_{v_1 \in \mathrm{Vert}(\Gamma_1)} \mathfrak{D}(f_1^{-1}(v_1)).$$

By our standard extension trick we may in fact assume that $\mathfrak{D}_k$ is defined on $\mathrm{Iso}_k\mathbf{\Gamma}(\!(g,S)\!)$ for an arbitrary finite set S. We will also assume that

$$\mathfrak{D}(*_{g,S}) = \mathbf{k}, \text{ for any } g, S. \tag{5.51}$$

Under this assumption, $\mathfrak{D}_1([\Gamma_0 \xrightarrow{f} \Gamma_1]) = \mathfrak{D}(\Gamma_1) \otimes \bigotimes_{v \in \mathrm{Vert}(\Gamma_1)} \mathfrak{D}(f^{-1}(v))$ is naturally isomorphic to $\mathfrak{D}(\Gamma_1)$,

$$\mathfrak{D}_1([\Gamma_0 \xrightarrow{f} \Gamma_1]) \cong \mathfrak{D}(\Gamma_1), \tag{5.52}$$

whenever f is an isomorphism.

Let $\partial_1 : \mathrm{Iso}_1\mathbf{\Gamma}(\!(g,S)\!) \to \mathrm{Iso}_0\mathbf{\Gamma}(\!(g,S)\!)$ be as in (5.32) and suppose we are given a transformation $\nu : \mathfrak{D}_1 \to \mathfrak{D} \circ \partial_1$. This means that to each $[\Gamma_0 \xrightarrow{f} \Gamma_1] \in \mathrm{Iso}_0\mathbf{\Gamma}(\!(g,S)\!)$ is assigned a natural morphism

$$\nu_f : \mathfrak{D}(\Gamma_1) \otimes \bigotimes_{v \in \mathrm{Vert}(\Gamma_1)} \mathfrak{D}(f^{-1}(v)) \longrightarrow \mathfrak{D}(\Gamma_0).$$

Such ν induces, for $1 \leq i \leq k$, transformations $\nu_{ki} : \mathfrak{D}_k \to \mathfrak{D}_{k-1} \circ \partial_i$ as follows. For $i = k$ and $\mathbf{\Gamma} := [\Gamma_0 \xrightarrow{f_1} \cdots \xrightarrow{f_k} \Gamma_k] \in \mathrm{Iso}_k\mathbf{\Gamma}(\!(g,S)\!)$, $(\nu_{kk})_{\mathbf{\Gamma}} : \mathfrak{D}_k(\mathbf{\Gamma}) \to (\mathfrak{D}_{k-1} \circ \partial_k)(\mathbf{\Gamma})$ is the composition

$$\mathfrak{D}_k(\mathbf{\Gamma}) = \mathfrak{D}(\Gamma_k) \otimes \bigotimes_{v_k \in \mathrm{Vert}(\Gamma_k)} \mathfrak{D}(f_k^{-1}(v_k)) \otimes \bigotimes_{v_{k-1} \in \mathrm{Vert}(\Gamma_{k-1})} \mathfrak{D}(f_{k-1}^{-1}(v_{k-1})) \otimes \cdots \otimes \bigotimes_{v_1 \in \mathrm{Vert}(\Gamma_1)} \mathfrak{D}(f_1^{-1}(v_1))$$
$$\xrightarrow{\nu_{f_k} \otimes \mathbb{1}} \mathfrak{D}(\Gamma_{k-1}) \otimes \bigotimes_{v_{k-1} \in \mathrm{Vert}(\Gamma_{k-1})} \mathfrak{D}(f_{k-1}^{-1}(v_{k-1})) \otimes \cdots \otimes \bigotimes_{v_1 \in \mathrm{Vert}(\Gamma_1)} \mathfrak{D}(f_1^{-1}(v_1)) = (\mathfrak{D}_{k-1} \circ \partial_k)(\mathbf{\Gamma}).$$

For $i < k$, $(\nu_{ki})_{\mathbf{\Gamma}} : \mathfrak{D}_k(\mathbf{\Gamma}) \to (\mathfrak{D}_{k-1} \circ \partial_i)(\mathbf{\Gamma})$ is given by

$$\begin{aligned}
\mathfrak{D}_k(\mathbf{\Gamma}) &= \mathfrak{D}(\Gamma_k)\otimes\cdots\otimes \bigotimes_{v_{i+1}\in \mathrm{Vert}(\Gamma_{i+1})} \mathfrak{D}(f_{i+1}^{-1}(v_{i+1}))\otimes \bigotimes_{v_i\in \mathrm{Vert}(\Gamma_i)} \mathfrak{D}(f_i^{-1}(v_i))\otimes\cdots\otimes \bigotimes_{v_1\in \mathrm{Vert}(\Gamma_1)} \mathfrak{D}(f_1^{-1}(v_1)) \\
&\cong \mathfrak{D}(\Gamma_k)\otimes\cdots\otimes \bigotimes_{v_{i+1}\in \mathrm{Vert}(\Gamma_{i+1})} [\mathfrak{D}(f_{i+1}^{-1}(v_{i+1}))\otimes \bigotimes_{v_i\in f_i^{-1}(v_{i+1})} \mathfrak{D}(f_{i+1}^{-1}(v_i))]\otimes\cdots\otimes \bigotimes_{v_1\in \mathrm{Vert}(\Gamma_1)} \mathfrak{D}(f_1^{-1}(v_1)) \\
&\xrightarrow{1\!\!1\otimes \bigotimes_{v_{i+1}} \nu_{f_i|f_{i+1}^{-1}(v_{i+1})}\otimes 1\!\!1} \mathfrak{D}(\Gamma_k)\otimes\cdots\otimes \bigotimes_{v_{i+1}\in \mathrm{Vert}(\Gamma_{i+1})} \mathfrak{D}((f_{i+1}f_i)^{-1}(v_{i+1}))\otimes\cdots\otimes \bigotimes_{v_1\in \mathrm{Vert}(\Gamma_1)} \mathfrak{D}(f_1^{-1}(v_1)) \\
&= (\mathfrak{D}_k \circ \partial_i)(\mathbf{\Gamma}).
\end{aligned}$$

Definition 5.3.11. *Let* $\mathfrak{D} : \mathrm{Iso}\,\mathbf{\Gamma}((g,n)) \to \mathtt{gVect}$ *and* $\nu : \mathfrak{D}_1 \to \mathfrak{D} \circ \partial_1$ *be as above. We call* $(\mathfrak{D}, \nu)$ *a* coefficient system, *(or a* hyperoperad *in the original terminology of [38]), if*

(i) the diagram

$$\begin{tikzcd}
\mathfrak{D}_2 \arrow[rr, "\nu_{22}"] \arrow[d, "\nu_{21}"'] & & \mathfrak{D}_1 \circ \partial_2 \arrow[d, "\nu \circ \partial_2"] \\
\mathfrak{D}_1 \circ \partial_1 \arrow[r, "\nu \circ \partial_1"] & \mathfrak{D}_1 \circ (\partial_1\partial_1) \arrow[r, phantom, "="] & \mathfrak{D}_1 \circ (\partial_1\partial_2)
\end{tikzcd}$$

commutes, and

(ii) for each isomorphism $f : \Gamma_1 \to \Gamma_0$, *the diagram*

$$\begin{tikzcd}
\mathfrak{D}_1([\Gamma_0 \xrightarrow{f} \Gamma_1]) \arrow[r, "\nu_f"] \arrow[dr, "\cong"] & \mathfrak{D}(\Gamma_0) \arrow[d, "\mathfrak{D}(f)"] \\
& \mathfrak{D}(\Gamma_1)
\end{tikzcd}$$

where $\cong$ *is the identification of (5.52), commutes.*

For a coefficient system $\mathfrak{D}$, define the twisted version $\mathbb{M}_{\mathfrak{D}}$ of the endofunctor $\mathbb{M}$ by

$$(\mathbb{M}_{\mathfrak{D}}\mathcal{E})((g,n)) := \operatorname*{colim}_{\Gamma \,\in\, \mathrm{Iso}\,\mathbf{\Gamma}((g,n))} \mathfrak{D}(\Gamma)\otimes\mathcal{E}((\Gamma)), \text{ for } \mathcal{E} \in \mathtt{MCol}.$$

The following proposition is an innocuous twisted version of Proposition 5.3.5 and we leave the proof to the reader.

Proposition 5.3.12. *For each* $k \geq 0$, *the* $(k+1)$*-th iterate of the functor* $\mathbb{M}_{\mathfrak{D}}$ *is given by*

$$(\mathbb{M}_{\mathfrak{D}})^{k+1}\mathcal{E}((g,S)) = \operatorname*{colim}_{[\Gamma_0 \xrightarrow{f_1} \ldots \xrightarrow{f_k} \Gamma_k] \,\in\, \mathrm{Iso}_k\mathbf{\Gamma}((g,S))} \mathcal{E}((\Gamma_0))\otimes\mathfrak{D}_k([\Gamma_0 \xrightarrow{f_1} \ldots \xrightarrow{f_k} \Gamma_k]). \qquad (5.53)$$

Using the description of (5.53), we may define a natural transformation $\mu : \mathbb{M}_{\mathfrak{D}}^2 \to \mathbb{M}_{\mathfrak{D}}$ by setting $\mu_{\mathcal{E}} : (\mathbb{M}_{\mathfrak{D}})^2\mathcal{E} \to \mathbb{M}_{\mathfrak{D}}\mathcal{E}$ to be the colimit of the system of maps

$$\mathcal{E}((\Gamma_0))\otimes\mathfrak{D}_1([\Gamma_0 \xrightarrow{f} \Gamma_1]) \xrightarrow{1\!\!1\otimes\nu_f} \mathcal{E}((\Gamma_0))\otimes\mathfrak{D}(\Gamma_0),\ [\Gamma_0 \xrightarrow{f} \Gamma_1] \in \mathrm{Iso}_1\mathbf{\Gamma}((g,S)).$$

Let also $v : 1\!\!1_{\mathtt{MCol}} \to \mathbb{M}_{\mathfrak{D}}$ be the transformation given by the inclusion of $\mathcal{E}((g,S))$ into the component $\mathcal{E}((*_{g,S}))\otimes\mathfrak{D}(*_{g,S}) \cong \mathcal{E}((g,S))$ of $(\mathbb{M}_{\mathfrak{D}}\mathcal{E})((g,S))$.

Theorem 5.3.13. *The transformations $\mu : \mathbb{M}_{\mathfrak{D}}^2 \to \mathbb{M}_{\mathfrak{D}}$ and $v : 1\!\!1_{\mathtt{MCol}} \to \mathbb{M}_{\mathfrak{D}}$ define on the functor $\mathbb{M}_{\mathfrak{D}}$ a structure of a triple.*

Proof. As in the proof of Theorem 5.3.6 we see that $\mu(\mu\mathbb{M}_{\mathfrak{D}})_{\mathcal{E}} : \mathbb{M}_{\mathfrak{D}}^3\mathcal{E} \to \mathbb{M}_{\mathfrak{D}}\mathcal{E}$ is the colimit, over $[\Gamma_0 \xrightarrow{f_1} \Gamma_1 \xrightarrow{f_2} \Gamma_2]$, of the system of maps

$$\mathcal{E}((\Gamma_0))\otimes\mathfrak{D}([\Gamma_0 \xrightarrow{f_1} \Gamma_1 \xrightarrow{f_2} \Gamma_2]) \xrightarrow{1\!\!1\otimes(\nu\circ\partial_2)\nu_{22}} \mathcal{E}((\Gamma_0))\otimes\mathfrak{D}(\Gamma_0),$$

while $\mu(\mathbb{M}_{\mathfrak{D}}\mu)_{\mathcal{E}} : \mathbb{M}_{\mathfrak{D}}^3\mathcal{E} \to \mathbb{M}_{\mathfrak{D}}\mathcal{E}$ is the colimit of

$$\mathcal{E}((\Gamma_0))\otimes\mathfrak{D}([\Gamma_0 \xrightarrow{f_1} \Gamma_1 \xrightarrow{f_2} \Gamma_2]) \xrightarrow{1\!\!1\otimes(\nu\circ\partial_1)\nu_{21}} \mathcal{E}((\Gamma_0))\otimes\mathfrak{D}(\Gamma_0).$$

Thus the first diagram of Definition 5.3.11 assures that $\mu(\mu\mathbb{M}_{\mathfrak{D}}) = \mu(\mathbb{M}_{\mathfrak{D}}\mu)$. The unitarity $\mu(v\mathbb{M}_{\mathfrak{D}}) = \mu(\mathbb{M}_{\mathfrak{D}}v) = \mathbb{M}_{\mathfrak{D}}$ is an easy consequence of the commutativity of the second diagram of Definition 5.3.11. ■

For two coefficient systems $\mathfrak{D} = (\mathfrak{D},\nu)$ and $\mathfrak{E} = (\mathfrak{E},\omega)$, define $\mathfrak{D}\otimes\mathfrak{E} = (\mathfrak{D}\otimes\mathfrak{E},\nu\otimes\omega)$ by $(\mathfrak{D}\otimes\mathfrak{E})(\Gamma) := \mathfrak{D}(\Gamma)\otimes\mathfrak{E}(\Gamma)$, with $(\nu\otimes\omega)_f : (\mathfrak{D}\otimes\mathfrak{E})_1([\Gamma_0 \xrightarrow{f} \Gamma_1]) \to (\mathfrak{D}\otimes\mathfrak{E})(\Gamma_0)$ given as the composition

$$\begin{aligned}(\mathfrak{D}\otimes\mathfrak{E})_1([\Gamma_0 \xrightarrow{f} \Gamma_1]) &= \mathfrak{D}(\Gamma_1)\otimes\mathfrak{E}(\Gamma_1)\otimes \bigotimes_{v\in\mathrm{Vert}(\Gamma_1)} \mathfrak{D}(f^{-1}(v))\otimes \bigotimes_{v\in\mathrm{Vert}(\Gamma_1)} \mathfrak{E}(f^{-1}(v))\\ &\cong \Big\{\mathfrak{D}(\Gamma_1)\otimes \bigotimes_{v\in\mathrm{Vert}(\Gamma_1)} \mathfrak{D}(f^{-1}(v))\Big\}\otimes\Big\{\mathfrak{E}(\Gamma_1)\otimes \bigotimes_{v\in\mathrm{Vert}(\Gamma_1)} \mathfrak{E}(f^{-1}(v))\Big\}\\ &\xrightarrow{\nu_f\otimes\omega_f} \mathfrak{D}(\Gamma_0)\otimes\mathfrak{E}(\Gamma_0) = (\mathfrak{D}\otimes\mathfrak{E})(\Gamma_0).\end{aligned}$$

Finally, let $\mathfrak{I} = (\mathfrak{I},\iota)$ be the coefficient system with $\mathfrak{I}(\Gamma) = \mathbf{k}$ for each $\Gamma \in \mathbf{\Gamma}((g,S))$ and ι given in the evident way.

Lemma 5.3.14. *The tensor product $\otimes$ introduced above defines a monoidal structure on the set of coefficient systems. The system $\mathfrak{I}$ is the unit for $\otimes$.*

The proof of the lemma is a direct verification and we leave it to the reader. Let us say that a graded vector space V is *invertible*, if there exists another graded vector space V^{-1} such that $V \otimes V^{-1} \cong \mathbf{k}$. Clearly, V is invertible if and only if it is of the form $\uparrow^n \mathbf{k}$ for some $n \in \mathbb{Z}$; then $V^{-1} = \downarrow^n \mathbf{k}$. According to [38] we call a coefficient system $\mathfrak{D} = (\mathfrak{D}, \nu)$ a *cocycle*, if $\mathfrak{D}(\Gamma)$ is invertible for each $\Gamma \in \mathrm{Iso}\,\mathbf{\Gamma}((g,S))$, and the maps ν_f are isomorphisms for each $f : \Gamma_0 \to \Gamma_1$. The inverse of a cocycle $\mathfrak{D}$ is again a cocycle which we denote $\mathfrak{D}^{-1}$.

A special class of cocycles can be constructed as follows. Let $\mathfrak{l}$ be a modular collection such that each graded vector space $\mathfrak{l}((g,n))$ is invertible. Define, for $\Gamma \in \mathrm{Iso}\,\mathbf{\Gamma}((g,S))$,

$$\mathfrak{D}_{\mathfrak{l}}(\Gamma) := \mathfrak{l}((g,S)) \otimes \bigotimes_{v \in \mathrm{Vert}(\Gamma)} \mathfrak{l}((g(v), \mathrm{Leg}(v)))^{-1}. \tag{5.54}$$

For $f : \Gamma_0 \to \Gamma_1$, let $\nu_{\mathfrak{l}} : \mathfrak{D}_{\mathfrak{l}}(\Gamma_1) \otimes \bigotimes_{v \in \mathrm{Vert}(\Gamma_1)} \mathfrak{D}_{\mathfrak{l}}(f^{-1}(v)) \to \mathfrak{D}_{\mathfrak{l}}(\Gamma_0)$ be given by the identification

$$\begin{aligned}
&\mathfrak{D}_{\mathfrak{l}}(\Gamma_1) \otimes \bigotimes_{v \in \mathrm{Vert}(\Gamma_1)} \mathfrak{D}_{\mathfrak{l}}(f^{-1}(v)) = \\
&\quad = \mathfrak{l}((g,S)) \otimes \bigotimes_{v \in \mathrm{Vert}(\Gamma_1)} \mathfrak{l}((g(v), \mathrm{Leg}(v)))^{-1} \otimes \Big\{ \bigotimes_{v \in \mathrm{Vert}(\Gamma_1)} \mathfrak{l}((g(v), \mathrm{Leg}(v))) \otimes \bigotimes_{w \in f^{-1}(v)} \mathfrak{l}((g(w), \mathrm{Leg}(w)))^{-1} \Big\} \\
&\quad \cong \mathfrak{l}((g,S)) \otimes \bigotimes_{v \in \mathrm{Vert}(\Gamma_1)} \Big\{ \mathfrak{l}((g(v), \mathrm{Leg}(v)))^{-1} \otimes \mathfrak{l}((g(v), \mathrm{Leg}(v))) \Big\} \otimes \bigotimes_{w \in \mathrm{Vert}(\Gamma_0)} \mathfrak{l}((g(w), \mathrm{Leg}(w)))^{-1} \\
&\quad \cong \mathfrak{l}((g,S)) \otimes \bigotimes_{w \in \mathrm{Vert}(\Gamma_0)} \mathfrak{l}((g(w), \mathrm{Leg}(w)))^{-1} = \mathfrak{D}_{\mathfrak{l}}(\Gamma_0).
\end{aligned}$$

Each collection $\mathfrak{l} \in \mathtt{MCol}$ defines an endofunctor $\mathfrak{l}(-) : \mathtt{MCol} \to \mathtt{MCol}$ by $\mathfrak{l}(\mathcal{E})((g,S)) := \mathfrak{l}((g,S)) \otimes \mathcal{E}((g,S))$.

Lemma 5.3.15. *The object $\mathfrak{D}_{\mathfrak{l}} = (\mathfrak{D}_{\mathfrak{l}}, \nu_{\mathfrak{l}})$ constructed above is a cocycle, called the* coboundary *of $\mathfrak{l}$. There is a natural isomorphism of triples*

$$\mathbb{M}_{\mathfrak{D} \otimes \mathfrak{D}_{\mathfrak{l}}} \cong \mathfrak{l} \circ \mathbb{M}_{\mathfrak{D}} \circ \mathfrak{l}^{-1}. \tag{5.55}$$

Proof. The verification that $\mathfrak{D}_{\mathfrak{l}}$ is a cocycle is easy and we leave it to the reader. The isomorphism (5.55) is given as the colimit, over $\Gamma \in \mathbf{\Gamma}((g,S))$, of the identifications

$$\begin{aligned}
&(\mathfrak{D} \otimes \mathfrak{D}_{\mathfrak{l}})(\Gamma) \otimes \mathcal{E}((\Gamma)) = \mathfrak{D}(\Gamma) \otimes \mathfrak{D}_{\mathfrak{l}}(\Gamma) \otimes \mathcal{E}((\Gamma)) \\
&\quad = \mathfrak{D}(\Gamma) \otimes \mathfrak{l}((g,S)) \otimes \bigotimes_{v \in \mathrm{Vert}(\Gamma)} \mathfrak{l}((g(v), \mathrm{Leg}(v)))^{-1} \otimes \mathcal{E}((\Gamma)) \cong \mathfrak{l}((g,S)) \otimes \Big\{ \mathfrak{D}(\Gamma) \otimes (\mathfrak{l}^{-1}\mathcal{E})((\Gamma)) \Big\}.
\end{aligned}$$

■

Example 5.3.16. Let $\mathfrak{s}(\!(g,n)\!) :=\uparrow^{2(g-1)+n+1}\mathrm{sgn}_{n+1}$, where sgn_{n+1} is the signum representation of Σ_{n+1}. For $\mathcal{E} \in \mathtt{MCol}$ we call $\mathfrak{s}\mathcal{E} = \mathfrak{s}\otimes\mathcal{E}$ the *modular suspension* of the collection $\mathcal{E}$. The associated cocycle $\mathfrak{D}_{\mathfrak{s}}$ is concentrated in degree zero. Indeed, we obtain from (5.54) that, for $\Gamma \in \mathbf{\Gamma}(\!(g,S)\!)$,

$$\deg(\mathfrak{D}_{\mathfrak{s}}(\Gamma)) = 2(g-1) + |S| - \sum_{v\in \mathrm{Vert}(\Gamma)} \Big(2(g(v)-1) + |\mathrm{Leg}(v)|\Big),$$

which is zero, by (5.28). Let us remark that the original definition of $\mathfrak{s}$ in [38] uses desuspension instead of suspension, $\mathfrak{s}(\!(g,n)\!) :=\downarrow^{2(g-1)+n+1}\mathrm{sgn}_{n+1}$.

For a finite-dimensional vector space V of dimension n, let

$$\mathrm{Det}(V) :=\downarrow^{n}\wedge^{n} V,$$

the one-dimensional top component of the n-fold exterior power of V concentrated in degree $-n$. If S is a finite set, define $\mathrm{Det}(S) := \mathrm{Det}(\mathbf{k}^S)$. Explicitly this means that, if $\mathrm{card}(S) = k$,

$$\mathrm{Det}(S) = \left(\bigoplus_{f:\{1,\ldots,k\}\cong S} \mathrm{Span}(\downarrow s_{f(1)} \wedge\cdots\wedge \downarrow s_{f(k)})\right)_{\Sigma_k},$$

with Σ_k acting by $\sigma(\downarrow s_{f(1)} \wedge\cdots\wedge \downarrow s_{f(k)}) := \mathrm{sgn}(\sigma)(\downarrow s_{f(\sigma(1))} \wedge\cdots\wedge \downarrow s_{f(\sigma(k))})$. The following lemma is an easy consequence of definitions.

Lemma 5.3.17. *For any finite set S, there is a natural isomorphism*

$$\mathrm{Det}(S)^2 \cong\downarrow^{2|S|}\mathbf{k}. \tag{5.56}$$

Given a decomposition $S = \bigsqcup_{i\in I} S_i$, of a finite set S into a disjoint union, then there is a natural identification

$$\mathrm{Det}(S) \cong \bigotimes_{i\in I} \mathrm{Det}(S_i). \tag{5.57}$$

Let us introduce the cocycle $\mathfrak{K} = (\mathfrak{K},\mathfrak{k})$ by $\mathfrak{K}(\Gamma) := \mathrm{Det}(\mathrm{Edg}(\Gamma))$. The structure morphism $\mathfrak{k}_f : \mathfrak{K}(\Gamma_1)\otimes\bigotimes_{v\in\mathrm{Vert}(\Gamma_1)}\mathfrak{K}(f^{-1}(v)) \to \mathfrak{K}(\Gamma_0)$ is, for $f : \Gamma_0 \to \Gamma_1$, given by the natural isomorphism

$$\begin{aligned}\mathfrak{K}(\Gamma_1)\otimes \bigotimes_{v\in\mathrm{Vert}(\Gamma_1)} \mathfrak{K}(f^{-1}(v)) &= \mathrm{Det}(\mathrm{Edg}(\Gamma_1))\otimes \bigotimes_{v\in\mathrm{Vert}(\Gamma_1)} \mathrm{Det}(\mathrm{Edg}(f^{-1}(v)))\\ &\cong \mathrm{Det}\Big(\mathrm{Edg}(\Gamma_1)\sqcup \bigsqcup_{v\in\mathrm{Vert}(\Gamma_1)} \mathrm{Edg}(f^{-1}(v))\Big) \cong \mathrm{Det}(\mathrm{Edg}(\Gamma_0)) = \mathfrak{K}(\mathrm{Edg}(\Gamma_0))\end{aligned}$$

based on (5.57). ■

We call $\mathfrak{K}$ the *dualizing cocycle*. Given a cocycle $\mathfrak{D}$, we denote

$$\check{\mathfrak{D}} := \mathfrak{K} \otimes \mathfrak{D}^{-1} \tag{5.58}$$

and call $\check{\mathfrak{D}}$ the *dual of* $\mathfrak{D}$. This duality will be crucial in the definition of the Feynman transform for modular operads.

Let $\mathrm{Or}(e)$ be the orientation line of an edge e in Γ, that is, the determinant $\uparrow^2 \mathrm{Det}(\{s,t\})$, where s and t are the pair of flags making up the edge e. The *orientation cocycle* $\mathfrak{T}$ of a graph Γ is given by

$$\mathfrak{T}(\Gamma) := \mathrm{Det}\Big(\bigoplus_{e \in \mathrm{Edg}(\Gamma)} \mathrm{Or}(e) \Big)$$

with the structure morphism $\mathfrak{t}_f : \mathfrak{T}(\Gamma_1) \otimes \bigotimes_{v \in \mathrm{Vert}(\Gamma_1)} \mathfrak{T}(f^{-1}(v)) \to \mathfrak{T}(\Gamma_0)$ given, for $f : \Gamma_0 \to \Gamma_1$, by the identification

$$\begin{aligned}
\mathfrak{T}(\Gamma_1) \otimes \bigotimes_{v \in \mathrm{Vert}(\Gamma_1)} \mathfrak{T}(f^{-1}(v)) &= \mathrm{Det}\Big(\bigoplus_{e \in \mathrm{Edg}(\Gamma_1)} \mathrm{Or}(e) \Big) \otimes \bigotimes_{v \in \mathrm{Vert}(\Gamma_1)} \Big(\bigoplus_{e \in \mathrm{Edg}(f^{-1}(v))} \mathrm{Or}(e) \Big) \\
&\cong \mathrm{Det}\Big(\bigoplus \{ \mathrm{Or}(e);\ e \in \mathrm{Edg}(\Gamma_1) \sqcup \bigsqcup_{v \in \mathrm{Vert}(\Gamma_1)} \mathrm{Edg}(f^{-1}(v)) \} \Big) \\
&\cong \mathrm{Det}\Big(\bigoplus_{e \in \mathrm{Edg}(\Gamma_0)} \mathrm{Or}(e) \Big) = \mathfrak{T}(\Gamma_0)
\end{aligned}$$

based on an obvious vector-space variant of (5.57) saying that, for a decomposition $V = \bigoplus_{i \in I} V_i$ of a finite-dimensional vector space,

$$\mathrm{Det}(V) \cong \bigotimes_{i \in I} \mathrm{Det}(V_i).$$

Proposition 5.3.18. *There is a natural isomorphism*

$$\mathfrak{D}_{\mathfrak{s}} \otimes \mathfrak{K} \cong \mathfrak{T},$$

where $\mathfrak{D}_{\mathfrak{s}}$ is the coboundary of the suspension $\mathfrak{s}$ introduced in Example 5.3.16.

Proof. For symbols x and y, let $\downarrow x \wedge \downarrow y$ denote the generator of $\mathrm{Det}(\mathrm{Span}\{x,y\})$. If s and t are two half-edges forming an edge $e \in \mathrm{Edg}(\Gamma)$, then the orientation line $\mathrm{Or}(e)$ is, by definition, spanned by $\uparrow^2 (\downarrow s \wedge \downarrow t)$, thus $\mathrm{Det}(\mathrm{Or}(e))$ is spanned by $\uparrow (\downarrow s \wedge \downarrow t)$, because the determinant of an one-dimensional space is just its desuspension. Let us identify $\uparrow(\downarrow s \wedge \downarrow t)$ with $\uparrow e \otimes (\downarrow s \wedge \downarrow t) \in (\mathrm{Det}\{e\})^{-1} \otimes \mathrm{Det}(\{s,t\})$. This means that for each $e = \{s,t\} \in \mathrm{Edg}(\Gamma)$, we defined a canonical identification

$$\mathrm{Det}(\mathrm{Or}(e)) \cong \mathrm{Det}(\{e\})^{-1} \otimes \mathrm{Det}(\{s,t\}), \tag{5.59}$$

compatible with the action of the automorphism group $\mathbb{Z}_2$ which interchanges s and t.

Let IFlag(Γ) denote the set of inner flags of the graph Γ, that is, the set of flags that are not the legs of Γ. Then, multiplying (5.59) over IFlag(Γ) and using (5.57), we derive that

$$\mathfrak{T}(\Gamma) \cong \mathrm{Det}(\mathrm{Edg}(\Gamma))^{-1} \otimes \mathrm{Det}(\mathrm{IFlag}(\Gamma)). \tag{5.60}$$

On the other hand, $\mathrm{Flag}(\Gamma) = \mathrm{IFlag}(\Gamma) \sqcup \mathrm{Leg}(\Gamma)$, thus (5.60) gives

$$\mathfrak{T}(\Gamma) \cong \mathrm{Det}(\mathrm{Edg}(\Gamma))^{-1} \otimes \mathrm{Det}(\mathrm{Flag}(\Gamma)) \otimes \mathrm{Det}(\mathrm{Leg}(\Gamma))^{-1}.$$

It remains to show that

$$\mathfrak{D}_{\mathfrak{s}}(\Gamma) \cong \mathrm{Det}(\mathrm{Edg}(\Gamma))^{-2} \otimes \mathrm{Det}(\mathrm{Flag}(\Gamma)) \otimes \mathrm{Det}(\mathrm{Leg}(\Gamma))^{-1} \tag{5.61}$$

which, because $\mathfrak{K}(\Gamma) = \mathrm{Det}(\mathrm{Edg}(\Gamma))$, implies the proposition. To prove (5.61), observe that, for $v \in \mathrm{Vert}(\Gamma)$,

$$\mathfrak{s}((g(v), \mathrm{Leg}(v))) \cong \uparrow^{2(g(v)-1)} \mathrm{Det}^{-1}(\mathrm{Leg}(v)).$$

Thus

$$\mathfrak{D}_{\mathfrak{s}}(\Gamma) \cong \uparrow^{2(g-1)} \mathrm{Det}^{-1}(\mathrm{Leg}(\Gamma)) \otimes \bigotimes_{v \in \mathrm{Vert}(\Gamma)} \downarrow^{2(g(v)-1)} \mathrm{Det}(\mathrm{Leg}(v)).$$

Because $\downarrow^{2(g(v)-1)}$ shifts the degree by an even number, we may move it to the left and write

$$\mathfrak{D}_{\mathfrak{s}}(\Gamma) = \uparrow^{2[g-1-\sum_{v \in \mathrm{Vert}(\Gamma)}(g(v)-1)]} \Big(\mathrm{Det}^{-1}(\mathrm{Leg}(\Gamma)) \otimes \bigotimes_{v \in \mathrm{Vert}(\Gamma)} \mathrm{Det}(\mathrm{Leg}(v)) \Big).$$

By (5.27), $g - 1 - \sum_{v \in \mathrm{Vert}(\Gamma)}(g(v) - 1) = |\mathrm{Edg}(\Gamma)|$, thus

$$\mathfrak{D}_{\mathfrak{s}}(\Gamma) = \uparrow^{2|\mathrm{Edg}(\Gamma)|} \Big(\mathrm{Det}^{-1}(\mathrm{Leg}(\Gamma)) \otimes \bigotimes_{v \in \mathrm{Vert}(\Gamma)} \mathrm{Det}(\mathrm{Leg}(v)) \Big).$$

This gives (5.61), because, by (5.56), $\mathrm{Det}(\mathrm{Edg}(\Gamma))^{-2} \cong \uparrow^{2|\mathrm{Edg}(\Gamma)|} \mathbf{k}$. ■

Let us recall that, by Theorem 5.3.7, modular operads are algebras over the triple $\mathbb{M} = (\mathbb{M}, \mu, v)$. 'Twisted' modular operads are defined in a similar way.

Definition 5.3.19. *Let* $\mathfrak{D}$ *be a coefficient system. A* modular $\mathfrak{D}$-operad *is an algebra over the triple* $\mathbb{M}_{\mathfrak{D}} = (\mathbb{M}_{\mathfrak{D}}, \mu, v)$.

Lemma 5.3.15 implies that, for any modular collection $\mathfrak{l}$, the functor $\mathfrak{l} : \mathtt{MCol} \to \mathtt{MCol}$ induces an equivalence between the category of modular $\mathfrak{D}$-operads and the category of modular $\mathfrak{D} \otimes \mathfrak{D}_{\mathfrak{l}}$-operads as follows. By definition, a $\mathfrak{D}$-modular operad structure on $\mathcal{A}$ is given by a map $\alpha : \mathbb{M}_{\mathfrak{D}} \mathcal{A} \to \mathcal{A}$, which in turn induces the map

$$\mathfrak{l}\alpha : (\mathfrak{l} \circ \mathbb{M}_{\mathfrak{D}})\mathcal{A} \to \mathfrak{l}\mathcal{A}.$$

But $(\mathfrak{l} \circ \mathbb{M}_{\mathfrak{D}})\mathcal{A} = (\mathfrak{l} \circ \mathbb{M}_{\mathfrak{D}} \circ \mathfrak{l}^{-1})\mathfrak{l}\mathcal{A}$ is, by (5.55), naturally isomorphic to $\mathbb{M}_{\mathfrak{D}\otimes\mathfrak{D}_{\mathfrak{l}}}$, thus $\mathfrak{l}\alpha$ may be interpreted as a map

$$\mathfrak{l}\alpha : \mathbb{M}_{\mathfrak{D}\otimes\mathfrak{D}_{\mathfrak{l}}}\mathfrak{l}\mathcal{A} \to \mathfrak{l}\mathcal{A}$$

that defines a $\mathfrak{D}\otimes\mathfrak{D}_{\mathfrak{l}}$-modular operad structure on $\mathfrak{l}\mathcal{A}$.

The above observation can be intuitively reformulated as saying that changing $\mathfrak{D}$ by a coboundary does not change the category of $\mathfrak{D}$-modular operads.

Modular $\mathfrak{I}$-operads are just ordinary modular operads, with the corresponding notion of modular algebras as representations in the modular endomorphisms operad, see Definition 5.3.10. Modular algebras exist also for $\mathfrak{K}$-modular operads, since there is the '$\mathfrak{K}$-endomorphism' operad, as shown in the following example.

Example 5.3.20. Let $V = (V, B)$ be a chain complex with a graded antisymmetric nondegenerate bilinear form $B : V\otimes V \to V$ of degree -1. Define the modular collection $End_V^{\mathfrak{K}}$ by

$$End_V^{\mathfrak{K}}(\!(g, S)\!) := V^{\otimes S},$$

for $g \geq 0$ and a finite set S. To define, for $\Gamma \in \mathbf{\Gamma}(\!(g, S)\!)$, the 'composition map'

$$\alpha_\Gamma^{\mathfrak{K}} : End_V^{\mathfrak{K}}(\!(\Gamma)\!)\otimes\mathfrak{K}(\Gamma) \to End_V^{\mathfrak{K}}(\!(g, S)\!)$$

choose labels s_e, t_e such that $e = \{s_e, t_e\}$ for each edge $e \in \mathrm{Edg}(\Gamma)$. Then $\alpha_\Gamma^{\mathfrak{K}}$ is the following composition:

$$\begin{aligned}
&End_V^{\mathfrak{K}}(\!(\Gamma)\!)\otimes\mathfrak{K}(\Gamma) \cong \\
&\quad\cong V^{\otimes \mathrm{Flag}(\Gamma)}\otimes\mathrm{Det}\Big(\bigoplus_{e\in\mathrm{Edg}(\Gamma)}\mathrm{Or}(e)\Big) \cong V^{\otimes\mathrm{Leg}(\Gamma)}\otimes\bigotimes_{e\in\mathrm{Edg}(\Gamma)}\Big(V^{\otimes\{s_e,t_e\}}\otimes\mathrm{Det}(\mathrm{Or}(e))\Big) \\
&\quad\cong V^{\otimes\mathrm{Leg}(\Gamma)}\otimes\bigotimes_{e\in\mathrm{Edg}(\Gamma)}\Big(V_{s_e}\otimes V_{t_e}\otimes\mathrm{Span}(\uparrow(\downarrow s_e\wedge\downarrow t_e))\Big) \\
&\qquad\xrightarrow{\ \mathbb{1}\otimes\bigotimes_e B_e\ } V^{\otimes\mathrm{Leg}(\Gamma)}\otimes\mathbf{k}^{\otimes\mathrm{Edg}(\Gamma)} \cong End_V^{\mathfrak{K}}(\!(g, S)\!),
\end{aligned}$$

where B_e is the map that sends $u\otimes v\otimes\uparrow(\downarrow s_e\wedge\downarrow t_e) \in V_{s_e}\otimes V_{t_e}\otimes\mathrm{Span}(\uparrow(\downarrow s_e\wedge\downarrow t_e))$ to $B(u, v) \in \mathbf{k}$.

We must show that the definition of $\alpha_\Gamma^{\mathfrak{K}}$ does not depend on the choice of labels. If we interchange labels s_e, t_e of an edge e, we obtain

$$(-1)^{|u|\cdot|v|+1}(v\otimes u\otimes\uparrow(\downarrow t_e\wedge\downarrow s_e)) \in V_{t_e}\otimes V_{s_e}\otimes\mathrm{Span}(\uparrow(\downarrow t_e\wedge\downarrow s_e))$$

instead of

$$u\otimes v\otimes\uparrow(\downarrow s_e\wedge\downarrow t_e) \in V_{s_e}\otimes V_{t_e}\otimes\mathrm{Span}(\uparrow(\downarrow s_e\wedge\downarrow t_e)).$$

The form B_e maps the element $(-1)^{|u|\cdot|v|+1}(v \otimes u \otimes \uparrow(\downarrow t_e \wedge \downarrow s_e))$ to $(-1)^{|u|\cdot|v|+1}B(v,u)$, which is the same as $B(u,v)$, because B is assumed to be graded antisymmetric. Thus the value of $\alpha^{\mathfrak{T}}_\Gamma$ does not depend on the choice of labels forming up the edge e. It is also clear that the degree of B_e is zero, thus also $\deg(\alpha^{\mathfrak{T}}_\Gamma) = 0$.

It is easy to verify that $\{\alpha^{\mathfrak{T}}_\Gamma | \ \Gamma \in \mathbf{\Gamma}((g,S))\}$ induces on $End^{\mathfrak{T}}_V$ the structure of a $\mathfrak{T}$-modular operad. See also a similar Example 5.6.4.

We are ready to define, for each cocycle $\mathfrak{D}$ and a $\mathfrak{D}$-modular operad $\mathcal{A}$, the Feynman transform $\mathsf{F}_{\mathfrak{D}}\mathcal{A} = (\mathsf{F}_{\mathfrak{D}}\mathcal{A}, \partial_{\mathsf{F}_{\mathfrak{D}}})$. Ignoring the differential, $\mathsf{F}_{\mathfrak{D}}\mathcal{A}$ equals to $\mathbb{M}_{\check{\mathfrak{D}}}\mathcal{A}^*$, the free $\check{\mathfrak{D}}$-modular operad on the linear dual $\mathcal{A}^*$ of $\mathcal{A}$, where $\check{\mathcal{D}}$ is the cocycle introduced in (5.58). The differential $\partial = \partial_{\mathsf{F}_{\mathfrak{D}}}$ is the sum $\partial_E + \partial_I$, where ∂_E (the exterior part) is induced in the standard way from the differential of $\mathcal{A}$, and ∂_I is defined as follows.

Since $\mathcal{A}$ is, by assumption, a $\mathfrak{D}$-modular operad, we are given, for each graph $\Gamma \in \mathbf{\Gamma}((g,S))$, the structure map $\alpha_\Gamma : \mathfrak{D}(\Gamma) \otimes \mathcal{A}((\Gamma)) \to \mathcal{A}((g(\Gamma), \mathrm{Leg}(\Gamma)))$. This structure map induces, for each $f : \Gamma_0 \to \Gamma_1$, a morphism (a 'twisted' analog of the morphism of (5.109))

$$\alpha((f)) : \mathfrak{D}(\Gamma_0) \otimes \mathcal{A}((\Gamma_0)) \to \mathfrak{D}(\Gamma_1) \otimes \mathcal{A}((\Gamma_1))$$

given as the composition

$$\begin{aligned}
\mathfrak{D}(\Gamma_0)\otimes\mathcal{A}((\Gamma_0)) &\xrightarrow{\nu_f^{-1}\otimes 1\!\!1} \mathfrak{D}(\Gamma_1)\otimes \bigotimes_{v\in \mathrm{Vert}(\Gamma_1)} \mathfrak{D}(f^{-1}(v))\otimes\mathcal{A}((\Gamma_0)) \\
&\cong \mathfrak{D}(\Gamma_1)\otimes \bigotimes_{v\in \mathrm{Vert}(\Gamma_1)} \Big(\mathfrak{D}(f^{-1}(v))\otimes\mathcal{A}((f^{-1}(v)))\Big) \\
&\xrightarrow{1\!\!1\otimes\bigotimes_v \nu_{f|f^{-1}(v)}} \mathfrak{D}(\Gamma_1)\otimes \bigotimes_{v\in \mathrm{Vert}(\Gamma_1)} \mathcal{A}((g(v), \mathrm{Leg}(v))) \cong \mathfrak{D}(\Gamma_1)\otimes\mathcal{A}((\Gamma_1)).
\end{aligned}$$

The above definition makes sense, since $\mathfrak{D}$ is, by assumption, a cocycle, thus the inverse of ν_f exists. The functoriality of the map $\alpha((-))$ is obvious. Let us denote

$$\alpha_{\Gamma,e} := \alpha((\pi_e)) : \mathfrak{D}(\Gamma)\otimes\mathcal{A}((\Gamma)) \longrightarrow \mathfrak{D}(\Gamma/e)\otimes\mathcal{A}((\Gamma/e)).$$

Using the identification $\mathfrak{D}(\Gamma)^* \cong \mathfrak{D}^{-1}(\Gamma)$, we may write the dual of $\alpha_{\Gamma,e}$ as

$$(\alpha_{\Gamma,e})^* : \mathfrak{D}(\Gamma/e)^{-1}\otimes\mathcal{A}((\Gamma/e))^* \longrightarrow \mathfrak{D}(\Gamma)^{-1}\otimes\mathcal{A}((\Gamma))^*.$$

There is another natural degree -1 map $\varepsilon_{\Gamma,e} : \mathfrak{K}(\Gamma/e) \to \mathfrak{K}(\Gamma)$ given by tensoring with the basic element $\downarrow e$ of $\mathfrak{K}(\{e\}) = \mathrm{Det}(\{e\})$ as

$$\begin{aligned}
\varepsilon_{\Gamma,e} : \mathfrak{K}(\Gamma/e) &\xrightarrow{\downarrow e\otimes 1\!\!1} \mathrm{Det}(\{e\})\otimes\mathfrak{K}(\Gamma/e) \\
&\cong \mathrm{Det}(\{e\})\otimes\mathrm{Det}(\mathrm{Edg}(\Gamma/e)) \cong \mathrm{Det}\Big(\{e\}\sqcup\mathrm{Edg}(\Gamma/e)\Big) \cong \mathrm{Det}(\mathrm{Edg}(\Gamma)) = \mathfrak{K}(\Gamma).
\end{aligned}$$

Let us denote the tensor product of $\varepsilon_{\Gamma,e}$ with the dual of $\alpha_{\Gamma,e}$ by $\partial_{\Gamma,e}$:

$$\begin{aligned}\partial_{\Gamma,e} : \check{\mathfrak{D}}(\Gamma/e)\otimes\mathcal{A}((\Gamma/e))^* &= \mathfrak{K}(\Gamma/e)\otimes\mathfrak{D}^{-1}(\Gamma/e)\otimes\mathcal{A}((\Gamma/e))^* \\ &\xrightarrow{\varepsilon_{\Gamma,e}\otimes(\alpha_{\Gamma,e})^*} \mathfrak{K}(\Gamma)\otimes\mathfrak{D}^{-1}(\Gamma)\otimes\mathcal{A}((\Gamma))^* = \check{\mathfrak{D}}(\Gamma)\otimes\mathcal{A}((\Gamma))^*.\end{aligned}$$

The vector space $\mathbb{M}_{\check{\mathfrak{D}}}\mathcal{A}((g,S))$ is the sum

$$\mathbb{M}_{\check{\mathfrak{D}}}\mathcal{A}((g,S)) \cong \bigoplus_{\gamma\in\{\mathbf{\Gamma}((g,S))\}} \left(\mathfrak{K}(\Gamma_\gamma)\otimes\mathfrak{D}^{-1}(\Gamma_\gamma)\otimes\mathcal{A}((\Gamma_\gamma))^*\right)_{\mathrm{Aut}(\Gamma_\gamma)} \tag{5.62}$$

over representatives of the isomorphism classes of elements of $\mathbf{\Gamma}((g,S))$ (compare formula (5.31) for $\mathbb{M}\mathcal{A}$). For two classes $\gamma,\delta\in\{\mathbf{\Gamma}((g,S))\}$, the 'matrix element'

$$\partial_{\gamma,\delta} : \mathfrak{K}(\Gamma_\gamma)\otimes\mathfrak{D}^{-1}(\Gamma_\gamma)\otimes\mathcal{A}((\Gamma_\gamma))^* \to \mathfrak{K}(\Gamma_\delta)\otimes\mathfrak{D}^{-1}(\Gamma_\delta)\otimes\mathcal{A}((\Gamma_\delta))^*$$

is defined as the summation of all $\partial_{\Gamma_\delta,e}$ over all edges e of Γ_δ such that $\Gamma_\gamma\cong\Gamma_\delta/e$. If no such an edge exists, we put $\partial_{\gamma,\delta}:=0$. Of course, $\partial_{\gamma,\delta}$ vanishes if $|\mathrm{Edg}(\Gamma_\gamma)\neq|\mathrm{Edg}(\Gamma_\delta)|-1$. The map $\partial_{\gamma,\delta}$ descends to the map (denoted by the same symbol)

$$\partial_{\gamma,\delta} : \left(\mathfrak{K}(\Gamma_\gamma)\otimes\mathfrak{D}^{-1}(\Gamma_\gamma)\otimes\mathcal{A}((\Gamma_\gamma))^*\right)_{\mathrm{Aut}(\Gamma_\gamma)} \to \left(\mathfrak{K}(\Gamma_\gamma)\otimes\mathfrak{D}^{-1}(\Gamma_\gamma)\otimes\mathcal{A}((\Gamma_\gamma))^*\right)_{\mathrm{Aut}(\Gamma_\delta)} \tag{5.63}$$

because any automorphism ϕ of $\Gamma_\gamma\cong\Gamma_\delta/e$ clearly lifts to some automorphism $\tilde\phi\in\mathrm{Aut}(\Gamma_\delta)$. The map in (5.63) also obviously does not depend on the choice of an isomorphism $\Gamma_\gamma\cong\Gamma_\delta/e$, because any two such isomorphisms differ by an automorphism of Γ_γ.

Finally, define ∂_I on the component $\left(\mathfrak{K}(\Gamma_\gamma)\otimes\mathfrak{D}^{-1}(\Gamma_\gamma)\otimes\mathcal{A}((\Gamma_\gamma))^*\right)_{\mathrm{Aut}(\Gamma_\gamma)}$ of (5.62) as the sum

$$\partial_I := \sum_{\delta\in\{\mathbf{\Gamma}((g,S))\}} \partial_{\gamma,\delta}.$$

This sum is finite, since, for each Γ_γ, there clearly exist only finitely many couples (δ,e), $\delta\in\{\mathbf{\Gamma}((g,S))\}$, $e\in\mathrm{Edg}(\Gamma_\delta)$, such that $\Gamma_\gamma\cong\Gamma_\delta/e$.

Theorem 5.3.21. *The object* $\mathsf{F}_{\mathfrak{D}}\mathcal{A}=(\mathbb{M}_{\check{\mathfrak{D}}}\mathcal{A}^*,\partial)$, *with* $\partial:=\partial_I+\partial_E$ *defined above, is a modular* $\check{\mathfrak{D}}$*-operad in the category of chain complexes. It is called the* Feynman transform *of the* $\mathfrak{D}$*-modular operad* $\mathcal{A}$.

Proof. The 'matrix element'

$$(\partial^2)_{\gamma,\omega} : \left(\mathfrak{K}(\Gamma_\gamma)\otimes\mathfrak{D}^{-1}(\Gamma_\gamma)\otimes\mathcal{A}((\Gamma_\gamma))^*\right)_{\mathrm{Aut}(\Gamma_\gamma)}) \longrightarrow \left(\mathfrak{K}(\Gamma_\omega)\otimes\mathfrak{D}^{-1}(\Gamma_\omega)\otimes\mathcal{A}((\Gamma_\omega))^*\right)_{\mathrm{Aut}(\Gamma_\omega)})$$

can be clearly written as

$$(\partial^2)_{\gamma,\omega} = \sum_{e_1,e_2}(\partial_{\Gamma_\omega,e_1}\circ\partial_{\Gamma_\omega/e_1,e_2}+\partial_{\Gamma_\omega,e_2}\circ\partial_{\Gamma_\omega/e_2,e_1})$$

where the summation is taken over pairs $\{e_1, e_2\}$ of distinct edges of Γ_ω such that $\Gamma_\gamma \cong \Gamma_\omega/\{e_1, e_2\}$. Thus, in order to prove that $\partial_I^2 = 0$, it is enough to show that

$$\partial_{\Gamma_\omega, e_1} \circ \partial_{\Gamma_\omega/e_1, e_2} = -\partial_{\Gamma_\omega, e_2} \circ \partial_{\Gamma_\omega/e_2, e_1} \tag{5.64}$$

which is, by definition, the same as to show that

$$\begin{aligned}\Big(\varepsilon_{\Gamma_\omega, e_1} \otimes (\alpha_{\Gamma_\omega, e_1})^*\Big) \circ \Big(\varepsilon_{\Gamma_\omega/e_1, e_2} \otimes (\alpha_{\Gamma_\omega/e_1, e_2})^*\Big) &= \\ = -\Big(\varepsilon_{\Gamma_\omega, e_2} \otimes (\alpha_{\Gamma_\omega, e_2})^*\Big) &\circ \Big(\varepsilon_{\Gamma_\omega/e_2, e_1} \otimes (\alpha_{\Gamma_\omega/e_2, e_1})^*\Big)\end{aligned} \tag{5.65}$$

The functoriality of $\alpha((-))$ implies that

$$\alpha_{\Gamma_\omega/e_1, e_2} \circ \alpha_{\Gamma_\omega, e_1} = \alpha_{\Gamma_\omega/e_2, e_1} \circ \alpha_{\Gamma_\omega, e_2}$$

thus (5.134) will follow from

$$\varepsilon_{\Gamma_\omega, e_1} \circ \varepsilon_{\Gamma_\omega/e_1, e_2} = -\varepsilon_{\Gamma_\omega, e_2} \circ \varepsilon_{\Gamma_\omega/e_2, e_1}. \tag{5.66}$$

By definition, $\varepsilon_{\Gamma_\omega, e_1} \circ \varepsilon_{\Gamma_\omega/e_1, e_2}$ maps $x \in \mathfrak{K}(\Gamma_\omega/\{e_1, e_2\})$ to

$$\downarrow e_1 \otimes \downarrow e_2 \otimes x \in \mathrm{Det}(\{e_1\}) \otimes \mathrm{Det}(\{e_2\}) \otimes \mathrm{Det}(\mathrm{Edg}(\Gamma_\omega/\{e_1, e_2\})) \cong \mathrm{Det}(\mathrm{Edg}(\Gamma_\omega)) = \mathfrak{K}(\Gamma_\omega).$$

Similarly, $\varepsilon_{\Gamma_\omega, e_2} \circ \varepsilon_{\Gamma_\omega/e_2, e_1}$ sends $x \in \mathfrak{K}(\Gamma_\omega/\{e_1, e_2\})$ to

$$\downarrow e_2 \otimes \downarrow e_1 \otimes x \in \mathrm{Det}(\{e_2\}) \otimes \mathrm{Det}(\{e_1\}) \otimes \mathrm{Det}(\mathrm{Edg}(\Gamma_\omega/\{e_1, e_2\})) \cong \mathrm{Det}(\mathrm{Edg}(\Gamma_\omega)) = \mathfrak{K}(\Gamma_\omega).$$

Equation (5.135) now follows from $\downarrow e_1 \otimes \downarrow e_2 = - \downarrow e_2 \otimes \downarrow e_1$. This also explains why we had to introduce the coefficient system $\mathfrak{K}$ into the definition of the Feynman transform.

Since all constructions involved in the definition of ∂_I were made in the category of graded differential vector spaces, ∂_I and ∂_E must commute as degree -1 objects, i.e. $\partial_I\partial_E + \partial_E\partial_I = 0$. Thus

$$\partial^2 = (\partial_I + \partial_E)^2 = \partial_I^2 + (\partial_I\partial_E + \partial_E\partial_I) + \partial_E^2 = 0.$$

The last thing that remains to be proved is that the differential ∂ is compatible with the $\check{\mathfrak{D}}$-operad structure on $\mathbb{M}_{\check{\mathfrak{D}}}$, that is, the triple multiplication $\mu_{\mathcal{A}^*} : (\mathbb{M}_{\check{\mathfrak{D}}})^2\mathcal{A}^* \to \mathbb{M}_{\check{\mathfrak{D}}}\mathcal{A}^*$ commutes with the differentials. The proof is quite straightforward.

By Proposition 5.3.12, $(\mathbb{M}_{\check{\mathfrak{D}}})^2\mathcal{A}^*((g, S))$ is the colimit

$$\operatorname*{colim}_{[\Gamma_0 \xrightarrow{f} \Gamma_1] \in \mathrm{Iso}_1\mathbf{\Gamma}((g,S))} \Big(\check{\mathfrak{D}}(\Gamma_1) \otimes \bigotimes_{v \in \mathrm{Vert}(\Gamma_1)} \check{\mathfrak{D}}(f^{-1}(v)) \otimes \mathcal{A}((\Gamma_0))^*\Big).$$

Because $\check{\mathfrak{D}}$ is a cocycle, $\check{\mathfrak{D}}(\Gamma_1) \otimes \bigotimes_{v \in \mathrm{Vert}(\Gamma_1)} \check{\mathfrak{D}}(f^{-1}(v))$ is *canonically and functorially* isomorphic, for each $[\Gamma_0 \xrightarrow{f} \Gamma_1] \in \mathrm{Iso}_1\mathbf{\Gamma}((g, S))$, to $\check{\mathfrak{D}}(\Gamma_0)$, thus the above colimit can be written as

$$(\mathbb{M}_{\check{\mathfrak{D}}})^2\mathcal{A}((g, S))^* = \operatorname*{colim}_{[\Gamma_0 \xrightarrow{f} \Gamma_1] \in \mathrm{Iso}_1\mathbf{\Gamma}((g,S))} \Big(\check{\mathfrak{D}}(\Gamma_0) \otimes \mathcal{A}((\Gamma_0))^*\Big).$$

The triple product to

$$\mathbb{M}_{\check{\mathfrak{D}}}\mathcal{A}((g,S))^* = \operatorname*{colim}_{[\Gamma_0]\,\in\,\mathrm{Iso}\,\mathbf{\Gamma}((g,S))} \Big(\check{\mathfrak{D}}(\Gamma_0)\otimes\mathcal{A}((\Gamma_0))^*\Big).$$

is then given by 'forgetting' the graph Γ_1 in the colimit. The statement is now almost tautological, because the definition of ∂ on $(\mathbb{M}_{\check{\mathfrak{D}}})^2\mathcal{A}^*$ does not involve any data related to the graph Γ_1. ∎

Let us state without proof, which can be found in [38], an important homotopy property of the Feynman transform. We call a morphism $f : \mathcal{A} \to \mathcal{B}$ of differential $\mathfrak{D}$-modular operads a *weak homotopy equivalence* if

$$f((g,n)) : \mathcal{A}((g,n)) \to \mathcal{B}((g,n))$$

is a homology isomorphism, for each g and n.

Proposition 5.3.22. *The Feynman transform is a homotopy functor. This means that, given a weak homotopy equivalence $f : \mathcal{A} \to \mathcal{B}$, the induced map $\mathsf{F}_{\mathfrak{D}}(f) : \mathsf{F}_{\mathfrak{D}}(\mathcal{A}) \to \mathsf{F}_{\mathfrak{D}}(\mathcal{B})$ is a also weak homotopy equivalence. Moreover, there is a canonical weak homotopy equivalence $\tau : \mathsf{F}_{\check{\mathfrak{D}}}\mathsf{F}_{\mathfrak{D}}(\mathcal{A}) \to \mathcal{A}$.*

5.4. Application: Graph Complexes.

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Let $F_{g,s}$ denote the genus g surface with s points removed, $g \geq 0$ and $s \geq 1$. We assume the stability condition $2g-2+s > 0$. Consider the Teichmüller space $\mathcal{T}_{g,s}$ of marked conformal classes of complete finite-area metrics on $F_{g,s}$. There is a fibration $\varphi : \mathcal{T}^{\mathrm{dec}}_{g,s} \to \mathcal{T}_{g,s}$, where the fiber over a point of $\mathcal{T}_{g,s}$ is the space of all horocycles about the punctures of $F_{g,s}$. The total space $\mathcal{T}^{\mathrm{dec}}_{g,s}$ of the fibration φ is called the *decorated Teichmüller space*. Since $\mathcal{T}_{g,s}$ is, by the uniformization theorem, the space of marked complete hyperbolic structures of finite area on $F_{g,s}$, the horocycles are determined by their lengths $\rho_i \in \mathbb{R}_+ := \{r \in \mathbb{R};\ r > 0\}$, $1 \leq i \leq s$, so that the fiber of φ is homeomorphic to $\mathbb{R}^s_+$. The dimension of $\mathcal{T}^{\mathrm{dec}}_{g,s}$ is easy to guess – there are $6g-6+2s$ Fenchel-Nielson coordinates for $\mathcal{T}_{g,s}$ plus s coordinates for the decorations, so the dimension must be $6g-6+3s$. In fact, the following more precise statement [97, Theorem 3.1] is true.

Theorem 5.4.1. *The decorated Teichmüller space $\mathcal{T}^{\mathrm{dec}}_{g,s}$ is homeomorphic to $\mathbb{R}^q_+$, where $q = 6g-6+3s$.*

The mapping class group $MC_{g,s}$ of isotopy classes of orientation-preserving homeomorphisms of $F_{g,s}$ (which may permute the punctures) acts in the natural way on $\mathcal{T}^{\rm dec}_{g,s}$ and the fibration φ is $MC_{g,s}$-equivariant. The moduli space $\mathcal{M}^{\rm dec}_{g,s} := \mathcal{T}^{\rm dec}_{g,s}/MC_{g,s}$ may be, according to the above remarks, understood as the moduli space of genus g Riemann surfaces with s distinct unlabeled punctures decorated by positive real numbers.

We are going to describe a neat combinatorial model for $\mathcal{M}^{\rm dec}_{g,s}$ whose existence follows from the results of [44] and [97]. We need first some terminology.

By a *ribbon graph* we mean a connected graph Γ with fixed cyclic orders on the set $\mathrm{edg}(v)$ of half-edges attached to each vertex $v \in \mathrm{Vert}(\Gamma)$. We assume that all vertices are at least ternary, that is, $|\mathrm{edg}(v)| \geq 3$ for each $v \in \mathrm{Vert}(\Gamma)$. We also assume that graphs considered in this section have no legs (i.e. all edges are inner), though this assumption can be relaxed at the cost of minor technical complications. One can associate an oriented surface with boundary, $\mathrm{Surf}(\Gamma)$, to each ribbon graph Γ by replacing edges by thin oriented rectangles (ribbons) and gluing them together at all vertices according to the chosen cyclic order. Let $RGr_{g,s}$ denote the set of all ribbon graphs Γ such that the surface $\mathrm{Surf}(\Gamma)$ has s holes and genus g.

A *metric* on the ribbon graph Γ is the map from the set of edges of Γ to the set of positive real numbers $\mathbb{R}_+$. We denote by $RGr^{\rm met}_{g,s}$ the set of ribbon graphs $\Gamma \in RGr_{g,s}$ with a metric. There is an obvious fibration of $RGr^{\rm met}_{g,s}$ over $RGr_{g,s}$; the fiber over a point $\Gamma \in RGr_{g,s}$ is a cell $\sigma_\Gamma \cong \mathbb{R}^k_+$, where k is the number of edges of Γ. Intuitively, when a length of an edge of Γ tends to zero, the graph degenerates to a new graph Γ', which determines a cell $\sigma_{\Gamma'}$ on the boundary of σ_Γ together with the gluing map. This gives $RGr^{\rm met}_{g,s}$ a structure of a cellular complex. Observe, however, that the closures of cells are not compact. Let us finally denote by $\mathcal{RG}^{\rm met}_{g,s}$ the corresponding moduli space. The following theorem follows from [97, Theorem 5.5].

Theorem 5.4.2. *The space $\mathcal{RG}^{\rm met}_{g,s}$ is, as an orbifold, canonically isomorphic to $\mathcal{M}^{\rm dec}_{g,s}$.*

We are going to describe an algebraic chain complex that calculates the (co)homology of $\mathcal{RG}^{\rm met}_{g,s}$. Our exposition follows [61, 62] where the details can be found.

The space $\mathcal{RG}^{\rm met}_{g,s}$ is a non-compact but smooth orbifold, with strata indexed by combinatorial types of underlying graphs. It is not oriented, but the orientation sheaf ϵ is easy to describe – its fiber over the metrized graph Γ is $\bigwedge^s(H^1(\Gamma;\mathbb{Q}))$. The Poincaré duality for (co)homology with closed supports [30, V.9.2] gives the isomorphisms

$$H^{\rm closed}_*(\mathcal{RG}^{\rm met}_{g,s};\epsilon) \cong H^{6g-6+3s-*}(\mathcal{RG}^{\rm met}_{g,s};\mathbb{Q}). \tag{5.67}$$

The homology $H^{\rm closed}_*(\mathcal{RG}^{\rm met}_{g,s};\epsilon)$ can be computed using a chain complex arising from the spectral sequence of the stratification. This chain complex is generated by graphs $\Gamma \in RGr_{g,s}$

endowed with an orientation that reflects the orientation of the corresponding strata. This orientation is given by ordering the vertices of Γ and assigning an orientation to each edge of Γ. Two orientations are assumed to be the same if they differ by an even number of changes. The orientation in the above sense of course differs from what is usually meant by orientation of a graph.

There are several equivalent ways to describe this orientation, we chosed that of [95] which gives the easiest description of the differential. Kontsevich defined an orientation of Γ as an orientation of the vector space $\mathbb{Q}^{\mathrm{edg}(\Gamma)} \oplus H_1(|\Gamma|;\mathbb{Q})$. To compare his definition with the definition above, consider the oriented cell complex of the CW-complex $|\Gamma|$, $\mathbb{Q}^{\mathrm{Vert}(\Gamma)} \longleftarrow \mathbb{Q}^{\mathrm{or}(\Gamma)}$, where $\mathrm{or}(\Gamma)$ is the set of oriented edges of Γ. It induces a long exact sequence

$$0 \longleftarrow \mathbb{Q} \longleftarrow \mathbb{Q}^{\mathrm{Vert}(\Gamma)} \longleftarrow \mathbb{Q}^{\mathrm{or}(\Gamma)} \longleftarrow H_1(|\Gamma|;\mathbb{Q}) \longleftarrow 0. \tag{5.68}$$

A moment's reflection convince us that the orientation of the vector space $\mathbb{Q}^{\mathrm{or}(\Gamma)}$ is given by specifying the orientation of edges *and* their order. The exact sequence (5.68) now immediately implies that both definitions of orientation agree.

Let us define the *ribbon graph complex* $RGC_k^{g,s}$ to be the free $\mathbb{Q}$-module generated by oriented ribbon graphs $\Gamma \in RGr_{g,s}$ with k edges subject the standard relation

$$\Gamma^- = -\Gamma, \tag{5.69}$$

where Γ^- is the graph Γ taken with the opposite orientation. The differential $\partial : RGC_{*+1}^{g,s} \to RGC_*^{g,s}$ is defined as follows.

For each $\Gamma \in RGr_{g,s}$ and an edge e joining two distinct vertices of Γ, let Γ/e be the graph obtained by collapsing e to a point. Observe that this new graph is again an element of $RGr_{g,s}$, because contracting e does not change the topology of $\mathrm{Surf}(\Gamma)$. Suppose that Γ is oriented in the above sense, with vertices numbered by $1,\ldots,m$, that is, $\mathrm{Vert}(\Gamma) = \{v_1,\ldots,v_m\}$. We define the orientation of Γ/e as follows. We may assume, applying a necessary number of changes, that the edge e points from v_{m-1} towards v_m. The set $\mathrm{Vert}(\Gamma/e)$ is isomorphic to $\{v_1,\ldots,v_{m-2},v_e\}$, where v_e is the vertex created by collapsing e. We assign to v_e label $m-1$ (thus the vertex with label m disappears). Since $\mathrm{edg}(\Gamma/e) \cong \mathrm{edg}(\Gamma) - \{e\}$, the orientation of edges of Γ induces an orientation of edges of Γ/e in an obvious manner. Now ∂ is given by

$$\partial(\Gamma) := \sum_e \Gamma/e,$$

the summation of oriented graphs Γ/e over all edges joining two distinct vertices of Γ. To prove that $\partial^2 = 0$ is an easy exercise and we can leave it to the reader.

The complex $RGC_*^{g,s} = (RGC_*^{g,s},\partial)$ calculates the closed-support homology of $\mathcal{RG}_{g,s}^{\mathrm{met}}$ with coefficients in ϵ, thus (5.67) gives

$$H_*(RGC_*^{g,s},\partial) \cong H^{6g-6+3s-*}(\mathcal{RG}_{g,s}^{\mathrm{met}};\mathbb{Q}).$$

Maxim Kontsevich showed [61, 62] that each strongly homotopy associative algebra with a scalar product **??** determines a class in the homology of $RGC^{g,s}_*$ and hence also in the rational cohomology of $\mathcal{M}^{\rm dec}_{g,s}$. The forgetful map $\mathcal{M}^{\rm dec}_{g,s} \to \mathcal{M}_{g,s}$ is clearly a rational homology isomorphism, he thus gave a method of constructing classes in the rational cohomology of the moduli space $\mathcal{M}_{g,s}$ itself. Recall also [44, p. 143] that $\mathcal{M}_{g,s}$ is 'rationally' the classifying space of the mapping class group,

$$H_*(\mathcal{M}_{g,s}; \mathbb{Q}) \cong H_*(BMC_{g,s}; \mathbb{Q}). \tag{5.70}$$

Another amazing property of the graph complex $RGC^{g,s}_*$ is its relation to the homology of a certain infinite dimensional Lie algebra. To be more precise, let a_n be the Lie algebra of derivations θ of the free associative algebra without unit on $p_1, \ldots, p_n, q_1, \ldots, q_n$ that satisfy the condition

$$\theta(\sum_{i=1}^{n} p_i q_i - q_i p_i) = 0,$$

and let $a_\infty := \varinjlim a_n$ be the direct limit. According to M. Kontsevich's interpretation, the algebra a_∞ is an algebra of hamiltonian vector fields on a flat symplectic infinite-dimensional manifold in the non-commutative geometry. The Chevalley-Eilenberg homology of this algebra has a primitively generated Hopf-algebra structure whose primitives $PH_*(a_\infty)$ can be described as [62, Theorem 1.1]:

$$PH_k(a_\infty) = PH_k(sp(\infty)) \oplus \bigoplus_{\substack{s>0 \\ 2g-2+s>0}} H^{2g+m-1+k}(RGC^{g,s}_*; \mathbb{Q}), \tag{5.71}$$

where the primitive cohomology of the infinite symplectic group $sp(\infty)$ is well-known:

$$PH_k(sp(\infty)) = \begin{cases} \mathbb{Q}, & \text{for } k \equiv 3 \bmod 4, \text{ and} \\ 0, & \text{otherwise.} \end{cases}$$

Let us remark that the $sp(\infty)$-part of (5.71) comes in fact from a graph complex, too. If we allow in the definition of $RGC^{g,s}_*$ also graphs with binary vertices, then the k-gon L_k generates $PH_k(sp(\infty))$. It is easy to verify that, for $k \not\equiv 3 \bmod 4$, there exists an automorphism of L_k inverting the orientation, consequently, by (5.69), $L_k = 0$ in $RGC^{1,2}_k$. This explains why the cohomology $PH_k(sp(\infty))$ is nontrivial only for $k \equiv 3 \bmod 4$.

Let us consider another version of the graph complex. Let $Gr^{(n)}_*$ denote the set of isomorphism classes of connected graphs Γ (no ribbon structure assumed) with all vertices at least ternary and Euler characteristic $\chi(\Gamma) = 1 - n$, graded by the number of vertices. The orientation of $\Gamma \in Gr^{(n)}_*$ is defined exactly as for ribbon graphs. Let $GC^{(n)}_*$ be the chain complex generated by all oriented graphs $\Gamma \in Gr^{(n)}$ modulo (5.69), with the differential given exactly as in the case of ribbon graphs.

The complex $GC^{(n)}_*$ calculates the closed homology of an orbifold $\mathcal{G}^{\rm met}_{(n)}$ of graphs $\Gamma \in Gr^{(n)}$ with a metric. Let us quote the following surprising fact on the topology of $\mathcal{G}^{\rm met}_{(n)}$ due to M. Culler and K. Vogtmann [19], which should be compared to (5.70).

Theorem 5.4.3. *The space $\mathcal{G}^{\rm met}_{(n)}$ is the classifying space of the group OutFree(n) of outer automorphisms of the free group with n generators.*

M. Kontsevich proved for the complex $GC^{(n)}_*$ a formula similar to (5.4.4), but with a more classical algebra c_∞ defined as the direct limit, $c_\infty = \varinjlim c_n$, of Lie algebras c_n of hamiltonian vector fields on the standard commutative symplectic plane $\mathbb{Q}[p_1,\ldots,p_n,q_1,\ldots,q_n]$ that preserve the origin $(p_1,\ldots,p_n,q_1,\ldots,q_n)$. Theorem 1.1 of [62] then states that

$$PH_k(c_\infty) = PH_k(sp(\infty)) \oplus \bigoplus_{n\geq 2} H_k(GC^{(n)}_*).$$

M. Kontsevich also suggested the third variant of the graph complex related to a hypothetic, supernatural Lie-algebra geometry.

The aim of this section is to show that the above graph complexes are all special cases of Feynman transform which we introduced in Section **??**. This fact was first observed in [38] and then made more precise in [84]. Recall **??** that each cyclic operad $\mathcal{P}$ can be considered as a modular operad (denoted by the same symbol)

$$\mathcal{P}((g,n+1)) = \begin{cases} 0, & \text{for } g>0 \text{ and} \\ \overline{\mathcal{P}}(n), & \text{for } g=0, \end{cases}$$

where $\overline{\mathcal{P}}$ is the 'augmentation ideal' of $\mathcal{P}$, see **??**. We can thus form its Feynman transform $\mathsf{F}\mathcal{P} = \{\mathsf{F}\mathcal{P}((g,n))\}$. We show that the complex $\mathsf{F}\mathcal{P}((g,0)) = (\mathsf{F}_*\mathcal{P}((g,0)), \partial_{\mathsf{F}})$ coincides, for $\mathcal{P} = \mathcal{A}ss$, $\mathcal{C}omm$ and $\mathcal{L}ie$, with the above mentioned graph complexes of M. Kontsevich. We then construct, in Section **??**, for each homotopy $\mathcal{P}^!$-algebra with inner product an element in the cohomology of $\mathsf{F}\mathcal{P}((g,0))$.

Let us look more closely at $\mathsf{F}\mathcal{P}((g,0))$. As a graded vector space, it is isomorphic to

$$\mathsf{M}_+(\uparrow\mathfrak{s}\overline{\mathcal{P}})((g,0)) = \bigoplus_{\Gamma\in \mathit{Graph}(g,0)} [\uparrow\mathfrak{s}\overline{\mathcal{P}}((\Gamma))]_{\mathrm{Aut}(\Gamma)},$$

where $\mathit{Graph}(g,0) := \{\Gamma\in \mathit{Graph}(0);\ \dim H_1(|\Gamma|;\mathbb{Q}) = g\}$. Recall that, in the above formula, $\uparrow$ denotes the vector space suspension, $\mathfrak{s}$ the operadic suspension **??** and M_+ the free modular-operad functor **??**. Formulas (**??**) and (**??**) give canonical identifications

$$\begin{aligned} \uparrow\mathfrak{s}\overline{\mathcal{P}}((\Gamma)) &\cong \uparrow^{|\mathrm{Vert}(\Gamma)|}[\det(\mathbf{k}^{\mathrm{Vert}(\Gamma)})\otimes\mathfrak{s}\overline{\mathcal{P}}((\Gamma))]_{\mathrm{Aut}(\Gamma)} \\ &\cong \uparrow^{|\mathrm{Vert}(\Gamma)|+2(g-1)}[\det(\mathbf{k}^{\mathrm{Vert}(\Gamma)})\otimes\det(\Gamma)\otimes\overline{\mathcal{P}}((\Gamma))]_{\mathrm{Aut}(\Gamma)}, \end{aligned} \tag{5.72}$$

where the determinant $\det(\Gamma)$ of the graph was defined in **??**. This suggests the following definition.

Definition 5.4.4. *A* $\mathcal{P}$-orientation *of a graph $\Gamma\in \mathit{Graph}(g,0)$ is given by*

(i) an order $\{v_1,\ldots,v_m\}$ of vertices of the graph Γ,

(ii) *an orientation of edges of* Γ, *and*
(iii) *an element* $p = p_1 \otimes \cdots \otimes p_m \in \bigotimes_{1 \leq i \leq m} \overline{\mathcal{P}}((v_i))$.

Two $\mathcal{P}$*-orientations are assumed to be the same if they differ by a sequence of the following two types of moves:*

(a) *We transpose the vertices* v_a *and* v_b, $1 \leq a < b \leq m$, *and change*

$$p = p_1 \otimes \cdots \otimes p_a \otimes \cdots \otimes p_b \otimes \cdots \otimes p_m$$

to

$$-(-1)^{|p_a||p_b|} p_1 \otimes \cdots \otimes p_b \otimes \cdots \otimes p_a \otimes \cdots \otimes p_m.$$

(b) *We change the orientation of an edge and multiply* p *by* -1.

We assign to the above $\mathcal{P}$*-orientation degree* $\sum_{1 \leq i \leq m}(|\mathrm{edg}(v_i)| + \deg(p_i) - 1)$.

The set of $\mathcal{P}$-or(Γ) of $\mathcal{P}$-orientations of Γ has an obvious structure of a graded $\mathbf{k}$-vector space. The following claim is not surprising.

Proposition 5.4.5. *The graded vector space* $\mathcal{P}$-or(Γ) *of* $\mathcal{P}$*-orientations of a graph* Γ *is canonically isomorphic to* $\uparrow \mathfrak{s}\overline{\mathcal{P}}((\Gamma))$.

Proof. The statement follows from (5.72) – the data (i) and (ii) of Definition 5.4.4 determine an element of $\det(\mathbf{k}^{\mathrm{Vert}(\Gamma)}) \otimes \det(\Gamma)$, while $p = p_1 \otimes \cdots \otimes p_m \in \bigotimes_{1 \leq i \leq m} \overline{\mathcal{P}}((v_i))$ determine an element of the unordered tensor product $\overline{\mathcal{P}}((\Gamma))$. ■

An immediate consequence of Proposition 5.4.5 is the isomorphism of graded vector spaces

$$\mathsf{F}\mathcal{P}((g,0)) \cong \bigoplus_{Graph(g,0)} [\mathcal{P}\text{-or}(\Gamma)]_{\mathrm{Aut}(\Gamma)} \tag{5.73}$$

Example 5.4.6. Let us demonstrate the use of (5.73) on the complex $\mathsf{F}Comm((g,0))$. Since $\mathcal{P}((S)) \cong \mathbf{k}$ for each finite set S, the element $p \in \bigotimes_{1 \leq i \leq m} \overline{\mathcal{P}}((v_i))$ of 5.4.4(iii) is just a scalar. This means that a *Comm*-orientation of a graph Γ is specified by a scalar $\alpha \in \mathbf{k}$, an order $\{v_1, \ldots, v_m\}$ of the vertices of Γ, and an orientation of the edges of Γ. To such a graph we assign degree $|\mathrm{Vert}(\Gamma)| + 2(g-1)$. Recalling the definition of the graph complex $Gr_*^{(n)}$ we conclude from (5.73), for $\mathbf{k} = \mathbb{Q}$ and $\mathcal{P} = Comm$, that

$$\mathsf{F}_{*-2n} Comm((1-n,0)) \cong Gr_*^{(n)}.$$

It is evident from definitions that, under the identification above, the differentials in $Gr_*^{(n)}$ and $\mathsf{F}_{*-2n} Comm((1-n,0))$ agree up to signs. The sign issue is a tiresome but straightforward exercise and we leave the verification to the reader.

Example 5.4.7. Consider the case $\mathcal{P} = Ass$. We show first that, for each finite set S with at least three elements, the space $Ass((S))$ is the $\mathbf{k}$-linear space span of the cyclic orders of the set S. Recall **??** that $Ass((n)) = \mathbf{k}[\Sigma_n]$, with the right Σ_{n+1}-action induced by the projection

$$\Sigma_{n+1} \longrightarrow C_{n+1}\backslash\Sigma_{n+1} \cong \Sigma_n \tag{5.74}$$

of Σ_{n+1} to the left classes of the action of the cyclic group C_{n+1} on Σ_{n+1}. From this we see that, for each nonzero element

$$p \in Ass((S)) = \left(\bigoplus_{f:S\to((n))} \mathbf{k}[\Sigma_n]\right)_{\Sigma_{n+1}} \tag{5.75}$$

there exist some indexing isomorphism $f : S \to ((n))$ such that the coordinate $p_f \in \mathbf{k}[\Sigma_n]_f$ of p is of the form $\alpha \cdot 1\!\!1_n$, $\alpha \in \mathbf{k}$. By (5.74), all maps f with this property differ by a composition with an element of C_{n+1}, they thus determine the same cyclic order on S.

On the other hand, given $\alpha \in \mathbf{k}$ and a cyclic order of S represented by an isomorphism $f : S \to ((n))$, the equivalence class of $p_f := 1\!\!1_n \in \mathbf{k}[\Sigma_n]_f$ in (5.75) does not depend on the choice of f and we may put $p := \alpha[1\!\!1 p_f]$.

We conclude that Ass-or(Γ) is the $\mathbf{k}$-linear span of isomorphism classes of ribbon graphs $\Gamma \in Graph(g, 0)$ with orientation given by an order of vertices and an orientation of edges. Formula (5.73) gives

$$\mathsf{F}_*Ass((g,0)) \cong RGC^{g,s}_{*+2(1-g)}.$$

Example 5.4.8. For $\mathcal{P} = Lie$, the Feynman transform $\mathsf{F}Lie((g,0))$ is the the graph complex whose definition is indicated in [62, S5]. We may imagine these 'Lie-graphs' as oriented graphs whose vertices are colored by $(k-1)!$ 'colors' representing a basis of $Lie(k)$, where $k+1$ is the number of edges at the vertex.

5.5. Application: Moduli spaces of surfaces of arbitrary genera.

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As we already mentioned at the beginning of Section **??**, modular operads were tailored to describe moduli spaces of surfaces of arbitrary genera. We illustrated this on the collection $\widehat{\mathcal{M}} = \{\widehat{\mathcal{M}}_{g,n}\}_{g\geq 0, n\geq 1}$ of Riemann surfaces with parameterized holes, and the obvious modular operad structure induced by sewing along the holes. This modular operad was studied by E. Getzler in [34, 35] who called algebras for this operad *topological conformal field theories* with trivial ghost number anomaly.

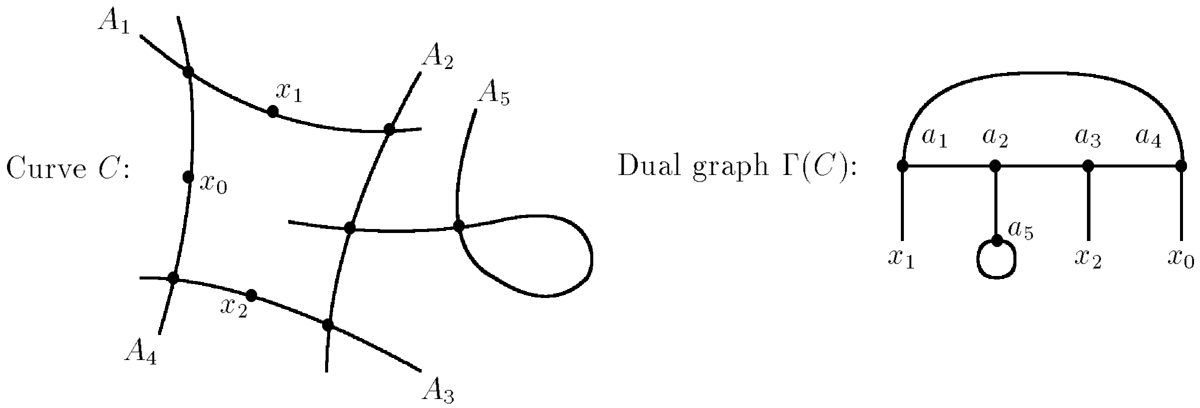

Figure 5.3: A stable curve and its dual graph. The curve C on the left has five components, A_1, A_2, A_3, A_4 and A_5, and three marked points x_0, x_1 and x_2. The dual graph $\Gamma(C)$ on the right has five vertices a_1, a_2, a_3, a_4 and a_5 corresponding to the components of the curve and three legs labeled by the marked points.

The aim of this section is to discuss less straightforward examples related to Deligne-Mumford moduli spaces $\overline{\mathcal{M}}_{g,n}$ of stable genus g Riemann surfaces with n punctures.

Deligne-Mumford moduli spaces. A *stable curve* with n-marked points (also called punctures), $n \geq 1$, is a connected complex projective curve C whose only singularities are ordinary double points (nodal singularities), together with a 'marking' given by an embedding of the set $\{x_1, \ldots, x_n\}$ in the set of smooth points of C. The stability means that we assume that there are no infinitesimal automorphisms of C fixing the marked points and double points. This is the same as to say that each smooth component of C isomorphic to the complex projective space $\mathbb{CP}^1$ has at least three special points, where by a special point we mean either a double point, or a singular point.

Observe that the stability puts no restriction on genus 1 components. Indeed, it follows from the connectivity of C and $n \geq 1$ that each component has at least one special point. In genus 1, one fixed point already excludes the existence of infinitesimal automorphisms.

The *dual graph* $\Gamma = \Gamma(C)$ of a stable curve $C = (C, x_1, \ldots, x_n)$ is a labeled graph whose vertices are the components of C, edges are the nodes, and its legs are the points $\{x_i\}_{1 \leq i \leq n}$. An edge e_y corresponding to a nodal point y joints the vertices corresponding to the components intersecting at y. The vertex v_K corresponding to a branch K is labeled by the genus of the normalization of K. (See [45, page 23] for the normalization and recall that a curve is normal if and only if it is nonsingular.)

The construction of $\Gamma(C)$ from a curve C is visualized on Figure 5.3. Let us denote by $\overline{\mathcal{M}}_{g,n}$ the coarse moduli space [45, page 347] of curves $C = (C, x_1, \ldots, x_n)$ such that the dual graph $\Gamma(C)$ has genus g (see **??** for the definition of the genus of a labeled graph).

Observe that the genus of the graph $\Gamma(C)$ equals the arithmetic genus of the curve C, $p_a(C) = g(\Gamma(C))$. This follows from the relation between the arithmetic genus of C and of its normalization [45, p. 298]. For example, the curve in Figure 5.3 has (arithmetic) genus $g(A_1)+g(A_2)+g(A_3)+g(A_4)+g(A_5)+2$, where $g(-)$ denotes the genus of the normalization of the corresponding component. Thus $\overline{\mathcal{M}}_{g,n}$ is the moduli space of stable curves with n marked points, of arithmetic genus g.

Let us observe that, for a curve $C \in \overline{\mathcal{M}}_{0,n}$, the graph $\Gamma(C)$ must necessarily be a tree, and all components of C must be smooth of genus 0, therefore $\overline{\mathcal{M}}_{0,n}$ coincides with the moduli space $\overline{\mathcal{M}}_n$ of genus 0 stable curves with n labeled punctures that we discussed in **??**. An immediate consequence of the stability is that dual graphs are stable as labeled graphs in the sense of **??**.

There is a subset $\mathcal{M}_{g,n}$ inside each $\overline{\mathcal{M}}_{g,n}$ consisting of curves C such that the dual graph $\Gamma(C)$ is the corolla with n legs, no edges and one vertex of genus g. By a result of P. Deligne, F.F. Knudsen and D. Mumford [23, 60, 59], $\overline{\mathcal{M}}_{g,n}$ is a projective variety obtained by adjoining to $\mathcal{M}_{g,n}$ a divisor $D_{g,n}$ with normal crossings, though one must interpret this result with care, since $\overline{\mathcal{M}}_{g,n}$ is, for $g \geq 1$, not smooth. We shall discuss this subtlety later in the section.

The symmetric group Σ_n acts on $\overline{\mathcal{M}}_{g,n}$ by renumbering the marked points. Therefore, with $\overline{\mathcal{M}}((g,n)) := \overline{\mathcal{M}}_{g,n+1}$,

$$\overline{\mathcal{M}} := \{\overline{\mathcal{M}}((g,n))\}_{g\geq 0, n\geq 1}$$

is a modular collection in the category of projective varieties. There are clearly no stable curves of genus g with n punctures if $2(g-1)+n \leq 0$, so $\overline{\mathcal{M}}$ is a stable modular collection in the sense of **??**.

Let us define the contraction along a stable graph $\Gamma \in \mathbf{\Gamma}((g,n))$

$$\alpha_\Gamma : \overline{\mathcal{M}}((\Gamma)) = \prod_{v\in \mathrm{Vert}(\Gamma)} \overline{\mathcal{M}}((g(v), \mathrm{Leg}(v))) \to \overline{\mathcal{M}}((g,n)), \tag{5.76}$$

by gluing the marked points of curves from $\overline{\mathcal{M}}((g(v), \mathrm{Leg}(v)))$, $v \in \mathrm{Vert}(\Gamma)$, according to the graph Γ. To be more precise, let

$$\prod_{v\in \mathrm{Vert}(\Gamma)} C_v,\ C_v \in \overline{\mathcal{M}}((g(v), \mathrm{Leg}(v))),$$

be an element of $\overline{\mathcal{M}}((\Gamma))$. Let e be an edge of the graph Γ connecting vertices v_1 and v_2, $e = \{y^e_{v_1}, y^e_{v_2}\}$, where $y^e_{v_i}$ is a marked point of the component C_{v_i}, $i = 1,2$, which is also the name of the corresponding flag of the graph Γ. The curve $\alpha_\Gamma(C)$ is then obtained by the identifications $y^e_{v_1} = y^e_{v_2}$ introducing a nodal singularity, for all $e \in \mathrm{Edg}(\Gamma)$. The procedure is the same as that described for the tree level in **??**. The contraction maps are iterations of operations studied in [59, page 190] and they are morphisms of projective varieties. The 'associativity' (**??**) of the contractions is clear immediately. Summing up the above observations, we formulate:

Theorem 5.5.1. *The contraction maps (5.76) define on the collection of coarse moduli spaces* $\overline{\mathcal{M}} = \{\overline{\mathcal{M}}((g,n))\}_{g\geq 0, n\geq 1}$ *the structure of a stable modular operad in the category of complex projective varieties.*

Passing to homology, we obtain a stable modular operad $H_*(\overline{\mathcal{M}})$ in the category of graded vector spaces. An algebra over this operad is, by definition, *a cohomological field theory* in the sense of M. Kontsevich and Yu. Manin [64], see also [35].

The contraction maps are closed imbeddings [59] and they describe the divisor $D((g,n)) := D_{g,n+1}$ as follows. For a stable graph $\Gamma \in \mathbf{\Gamma}((g,n))$, denote $D_\Gamma := \mathrm{Im}(\alpha_\Gamma)$. Then the union

$$D((g,n)) = \bigcup_{\Gamma} D_\Gamma, \tag{5.77}$$

over all stable graphs $\Gamma \in \mathbf{\Gamma}((g,n))$ with exactly one edge is a decomposition of $D((g,n))$ into smooth irreducible components. The same union as (5.77), but taken over graphs Γ with exactly two edges gives the locus of double points of $D((g,n))$, etc. In fact, (5.77) describes a decomposition of $D((g,n))$ into *closed* strata, compare **??**.

Let us turn out attention to open parts $\mathcal{M}_{g,n} \subset \overline{\mathcal{M}}_{g,n}$. The collection

$$\mathcal{M} = \{\mathcal{M}((g,n))\}_{g\geq 0, n\geq 1} \text{ with } \mathcal{M}((g,n)) := \mathcal{M}_{g,n+1},$$

is not a suboperad of $\overline{\mathcal{M}}$, because the images of the contraction maps (5.76) are contained in the divisor $D((g,n))$ while $D((g,n)) \cap \mathcal{M}((g,n)) = \emptyset$

The aim of the rest of this section is to show, following again [38], that suitable 'chains' on $\mathcal{M}((g,n))$ nevertheless form a certain $\mathfrak{K}$-modular operad, where $\mathfrak{K}$ is the dualizing cocycle introduced in **??**. By chains we mean the topological dual

$$\mathcal{E}^*(\overline{\mathcal{M}}((g,n)), \log D((g,n)))' \tag{5.78}$$

of the deRham complex $\mathcal{E}^*(\overline{\mathcal{M}}((g,n)), \log D((g,n)))$ of smooth differential forms on the variety $\overline{\mathcal{M}}((g,n))$ with logarithmic singularities on the divisor $D((g,n))$. Notice we used $(-)'$ to distinguish this topological dual from the algebraic one $(-)^*$. The precise meaning of this complex is explained below. Its homology is, by [46, Theorem 2.3], isomorphic to the Čech homology of $\mathcal{M}((g,n))$ with real coefficients,

$$\check{H}_*(\mathcal{M}((g,n))) \cong H_*(\mathcal{E}^*(\overline{\mathcal{M}}((g,n)), \log D((g,n)))'),$$

so (5.78) indeed forms a kind of 'chains' on $\mathcal{M}((g,n))$. Let us start with the necessary preliminary material.

Singularities and residua. The theory of deRham forms with logarithmic singularities stems from the following result of [46]. Let X be a smooth complex variety of dimension m and U an open (in the complex-analytic topology) subset of X. Let ζ and θ be smooth differential forms with compact support on U of dimension $2m$ and $2m-1$, respectively. Let φ be an arbitrary holomorphic function on U. Then the limits

$$\lim_{\delta\to 0}\int_{|\varphi|>\delta}\frac{\zeta}{\varphi} \quad\text{and}\quad \lim_{\delta\to 0}\int_{|\varphi|=\delta}\frac{\theta}{\varphi} \tag{5.79}$$

exist. Let us point out that we do not assume the forms ζ and θ to be holomorphic.

Suppose that $D\subset X$ is a divisor. Let us denote by $\mathcal{E}^*_X(\log D)$ the sheaf on X generated, as a sheaf of algebras, by deRham forms $\mathcal{E}^*_X$ on X and expressions $d\varphi/\varphi$, where φ is a holomorphic equation of D. A more formal way to introduce $\mathcal{E}^*_X(\log D)$ is the following. Let ι be the inclusion $\iota : X-D\hookrightarrow X$ and $\iota_*\mathcal{E}^*_{X-D}$ the direct image of the sheaf $\mathcal{E}^*_{X-D}$ of deRham forms on $X-D$. The sections over an open set $U\subset X$ are, by definition, given as

$$\iota_*\mathcal{E}^*_{X-D}(U) = \mathcal{E}^*(U-(U\cap D)).$$

The restriction

$$r:\mathcal{E}^*(U)\to\mathcal{E}^*(U-(U\cap D))$$

is clearly monic and it identifies $\mathcal{E}^*_X$ with a subsheaf of $\iota_*\mathcal{E}^*_{X-D}$. Observe also that if φ is a holomorphic equation of D in U, then the 1-form $d\varphi/\varphi$ can be considered as an element of $\mathcal{E}^*(U-(U\cap D))$. We may then define $\mathcal{E}^*_X(\log D)$ as the subsheaf of $\iota_*\mathcal{E}^*_{X-D}$ generated as a sheaf of subalgebras by $\mathcal{E}^*_X$ and 1-forms $d\varphi/\varphi$, where φ is a local holomorphic equation of D.

Let us define the *principal value* of a compactly supported form $\omega\in\mathcal{E}^{2m}(U,\log D)$ to be the integral

$$\mathrm{pv}\int_U\omega := \lim_{\delta\to 0}\int_{|\varphi|>\delta}\omega, \tag{5.80}$$

where φ is as above a holomorphic equation of D in U. The existence of this limit again follows from the results of [46]; it is more or less the first limit of (5.79). When $D\subset X$ is empty, $\mathcal{E}^*_X(\log D)=\mathcal{E}^*_X$ and the principal value is the ordinary integral over U.

Let us recall that, given an open subset U of a smooth manifold X, the space of smooth functions $\mathcal{C}^{(\infty)}(U)$ with the topology given by seminorms of the maxima of partial derivatives on compact subsets of U is a nuclear Fréchet linear topological space. The category `NF` of nuclear Fréchet spaces is a strict symmetric monoidal category, with the monoidal structure given by a certain completion $\widehat{\otimes}$ of the tensor product $\otimes$. A miraculous property of this topology is that, for any two open subsets $U_1\subset X_1$ and $U_2\subset X_2$, there exists a natural isomorphism

$$\mathcal{C}^{(\infty)}(U_1\times U_2)\xrightarrow{\cong}\mathcal{C}^{(\infty)}(U_1)\widehat{\otimes}\,\mathcal{C}^{(\infty)}(U_2)$$

of nuclear Fréchet spaces, see [42] or a bit easier [53, page 500]. The above isomorphism of course induces an isomorphism

$$Z:\mathcal{C}^{(\infty)}_{X_1\times X_2}\to\mathcal{C}^{(\infty)}_{X_1}\widehat{\otimes}\,\mathcal{C}^{(\infty)}_{X_2} \tag{5.81}$$

of sheaves of smooth functions, where the tensor product on the right is an obvious exterior product induced by the $\widehat{\otimes}$-product of NF-spaces.

Let us return back to our case of deRham forms with logarithmic singularities. Suppose that X_1, X_2 are smooth manifolds and $D_1 \subset X_1$, $D_2 \subset X_2$ divisors. Then

$$(D_1 \times X_1) \cup (X_1 \times D_2)$$

is a divisor in $X_1 \times X_2$ and the isomorphism (5.81) generalizes to

$$Z : \mathcal{E}^*_{X_1 \times X_2}(\log D_1 \times X_1 \cup X_1 \times D_2) \xrightarrow{\cong} \mathcal{E}_{X_1}(\log D_1) \widehat{\otimes} \mathcal{E}_{X_2}(\log D_2). \tag{5.82}$$

Suppose for a moment that D has no crossings, i.e. it that is a disjoint union of nonsingular components. Let $j : D \hookrightarrow X$ be the inclusion and $j_* \mathcal{E}^*_D$ the direct image of the sheaf $\mathcal{E}^*_D$. This means, of course, that

$$j_* \mathcal{E}^*_D(U) = \mathcal{E}^*(D \cap U), \tag{5.83}$$

for any open subset $U \subset X$. A result of [20] says that there exists a degree -1 map of graded sheaves

$$\mathrm{Res} : \mathcal{E}^*_X(\log D) \to j_* \mathcal{E}^{*-1}_D \tag{5.84}$$

such that for each open $U \subset X$ the induced map of sections $\mathrm{Res} : \mathcal{E}^*(U, \log D) \to \mathcal{E}^{*-1}(D \cap U)$ satisfies

$$\lim_{\delta \to 0} \frac{1}{2\pi i} \int_{|\varphi| = \delta} \omega\alpha = \int_D \mathrm{Res}(\omega)\alpha, \tag{5.85}$$

for each l-form $\omega \in \mathcal{E}^l(U, \log D)$, for each compactly supported form $\alpha \in \mathcal{E}^{2m-l-1}(U)$ and for each holomorphic equation φ of D in U.

Example 5.5.2. Formula (5.85) is a deep generalization of the residue theorem from complex analysis. If $X = \mathbb{C}$ is the complex plane, $D = \{0\}$ the origin interpreted as the divisor given by the equation $z = 0$ and U an open neighborhood of the origin, then $\mathcal{E}^1(U, \log D)$ consists of expressions $\omega = \omega_1 + \omega_2 \frac{dz}{z}$, where $\omega_1 \in \mathcal{E}^1(U)$, $\omega_2 \in \mathcal{E}^0(U)$ are smooth forms. The residue $\mathrm{Res} : \mathcal{E}^1(U) \to j_* \mathcal{E}^0(\{0\}) = \mathbb{C}$ is given by the classical formula $\mathrm{Res}(\omega) := \omega_2(0)$. Equation (5.5) translates to an easy consequence of the classical residue theorem,

$$\lim_{\delta \to 0} \frac{1}{2\pi i} \int_{S_\delta} \omega\alpha = \mathrm{Res}(\omega)\alpha(0), \tag{5.86}$$

where S_δ is the circle around the origin with diameter δ. Observe that we do not assume ω to be meromorphic or α holomorphic.

The situation is more complicated when the divisor D has crossings. It is then necessary to consider the normalization $\pi : \widetilde{D} \to D$. The inverse image $\widehat{D^2}$ of the locus of double points of D is a divisor with normal crossings in $\widetilde{D}$. Let j be as in (5.83) the inclusion $D \hookrightarrow X$. The residue is then the map

$$\mathrm{Res} : \mathcal{E}^*_X(\log D) \to (j\pi)_* \mathcal{E}^{*-1}_{\widetilde{D}}(\log \widehat{D^2}) \tag{5.87}$$

characterized as follows. For each open $U \subset X$ and for each l-form $\omega \in \mathcal{E}^l(U, \log D)$ its residue $\mathrm{Res}(\omega) \in (j\pi)_* \mathcal{E}_{\widehat{D}}(\log \widehat{D^2})(U) = \mathcal{E}^{l-1}(\pi^{-1}(U), \log \widehat{D^2})$ satisfies

$$\lim_{\delta \to 0} \frac{1}{2\pi i} \int_{|\varphi|=\delta} \omega\alpha = \mathrm{pv} \int_{\widetilde{D}} \mathrm{Res}(\omega)\pi^*\alpha,$$

for each compactly supported $\alpha \in \mathcal{E}^{2m-l-1}(U)$.

Let us analyze (5.87) in more details. The normalization $\widetilde{D}$ is the disjoint union

$$\widetilde{D} = \coprod_{1 \le i \le s} \widetilde{D}_i,$$

where each $\widetilde{D}_i$ is isomorphic to an irreducible component D_i of D. Let $\pi_i := \pi|_{\widetilde{D}_i}$, $D^2_i := D^2 \cap D_i$ and $\widehat{D^2_i} := \pi^{-1}(D^2_i) = \widehat{D^2} \cap \widetilde{D}_i$. From the isomorphism

$$(j\pi)_* \mathcal{E}^*_{\widetilde{D}}(\log \widehat{D^2}) \cong \bigoplus_{1 \le i \le s} (j\pi_i)_* \mathcal{E}^*_{\widetilde{D}_i}(\log \widehat{D^2_i})$$

and the isomorphism

$$(j\pi_i)_* \mathcal{E}^*_{\widetilde{D}_i}(\log \widehat{D^2_i}) \cong j_* \mathcal{E}^*_{D_i}(\log D^2_i),\ 1 \le i \le s,$$

induced by $\pi_i : \widetilde{D}_i \xrightarrow{\cong} D_i$, it follows that the residue (5.87) is given by an s-tuple of maps

$$\mathrm{Res} : \mathcal{E}^*_X(\log D) \to j_* \mathcal{E}^{*-1}_{D_i}(\log D^2_i),\ 1 \le i \le s. \tag{5.88}$$

For sections over X these maps give

$$\mathrm{Res}_i : \mathcal{E}^*(X, \log D) \to \mathcal{E}^{*-1}(D_i, \log D^2_i). \tag{5.89}$$

Example 5.5.3. Let us illustrate the above material on $X := \mathbb{C}^2$ and $D := \{z_1 z_2 = 0\}$. The divisor D decomposes as $D = D_1 \cup D_2$ with $D_1 = \{z_1 = 0\}$, $D_2 = \{z_2 = 0\}$, and the locus of double points is $D^2 = D_1 \cap D_2 = \{(0,0)\}$. System (5.89) for this special case is

$$\mathrm{Res}_i : \mathcal{E}^*(\mathbb{C}^2, \log D) \to \mathcal{E}^{*-1}(D_i, \log\{0\}),\ i = 1, 2.$$

Similarly, we have the residua for $(D_i, \{0\})$:

$$\mathrm{Res} : \mathcal{E}^1(D_i, \log\{0\}) \to \mathcal{E}^0(\{0\}) = \mathbb{C},\ i = 1, 2.$$

The above maps form the diagram

$$\begin{tikzcd}[column sep=small]
 & \mathcal{E}^1(D_1, \log\{0\}) \arrow[dr, "\mathrm{Res}"] & \\
\mathcal{E}^2(\mathbb{C}^2, \log(D_1 \cup D_2)) \arrow[ur, "\mathrm{Res}_1"] \arrow[dr, "\mathrm{Res}_2"'] & & \mathbb{C} \\
 & \mathcal{E}^1(D_2, \log\{0\}) \arrow[ur, "\mathrm{Res}"'] &
\end{tikzcd}$$

We claim the above diagram is *anticommutative*. This can be verified by a direct calculation, which we recommend as a nice exercise, or by the following argument.

The 'upper' composition U differs from the lower one L by the exchange $z_1 \leftrightarrow z_2$ or, formally, $U = L \circ T^*$, where $T : \mathbb{C}^2 \to \mathbb{C}^2$ is the flip $T(u \otimes v) := v \otimes u$. The anticommutativity now follows from
$$T^* \left(\frac{dz_1}{z_1} \frac{dz_2}{z_2} \right) = \frac{dz_2}{z_2} \frac{dz_1}{z_1} = -\frac{dz_1}{z_1} \frac{dz_2}{z_2}.$$

Operads $\mathcal{G}rav$, varG and Grav. Let us ignore for the moment the unpleasant fact that $\overline{\mathcal{M}}_{g,n}$ is not smooth and apply the above machinery to the divisor $D((g,n))$ in $\overline{\mathcal{M}}((g,n))$.

As in (5.77), the components D_Γ of the divisor $D((g,n))$ are indexed by stable modular graphs Γ with exactly one edge. We also observed that $D^2((g,n))$ (the locus of double points) is the union (5.77) over graphs $\Gamma \in \mathbf{\Gamma}((g,n))$ with exactly two inner edges, therefore
$$D^2_\Gamma := D^2((g,n)) \cap D_\Gamma = \bigcup_{\Delta < \Gamma} D_\Delta,$$
where the union runs over all graphs Δ, with exactly two edges. The relation $\Delta < \Gamma$ introduced in (**??**) means that the graph Γ is a contraction of Δ.

System (5.89) in this case gives, for each $\Gamma \in \mathbf{\Gamma}((g,n))$ with exactly one edge, the residue
$$\mathrm{Res}_\Gamma : \mathcal{E}^*(\overline{\mathcal{M}}((g,n)), \log D((g,n))) \to \mathcal{E}^{*-1}(\overline{\mathcal{M}}((\Gamma)), \log D^2_\Gamma).$$

Let us denote by $\mathsf{varG} := \{\mathsf{varG}((g,n))\}_{g \geq 0, n \geq 1}$ the modular collection in the monoidal category `Chain(NF)` of chain complexes of Fréchet nuclear spaces with
$$\mathsf{varG}((g,n)) := \mathcal{E}^*(\overline{\mathcal{M}}((g,n)), \log D((g,n)));$$
the meaning of this strange notation will become clear later. We claim that the isomorphism Z of (5.82) defines, for any $\Gamma \in \mathbf{\Gamma}((g,n))$ with exactly one edge, an isomorphism
$$Z_\Gamma : \mathcal{E}^*(\overline{\mathcal{M}}((\Gamma)), \log D^2_\Gamma) \xrightarrow{\cong} \mathsf{varG}((\Gamma)).$$

Indeed, if the unique edge of Γ joints two distinct vertices, then Γ is obtained from two corollae $*_1 \in \mathbf{\Gamma}((g_1, l))$, $*_2 \in \mathbf{\Gamma}((g_2, k))$, $g_1 + g_2 = g$, $k + l = n - 1$, by joining a leg of $*_1$ with a leg of $*_2$. Then clearly $\overline{\mathcal{M}}((\Gamma)) \cong \overline{\mathcal{M}}((*_1)) \times \overline{\mathcal{M}}((*_2))$ and
$$D^2_\Gamma \cong (\overline{\mathcal{M}}((*_1)) \times D_{*_2}) \cup (D_{*_1} \times \overline{\mathcal{M}}((*_2))),$$

therefore

$$\mathcal{E}^*(\overline{\mathcal{M}}((\Gamma)), \log D^2_\Gamma) \cong \mathcal{E}^*(\overline{\mathcal{M}}((\xi_1)), \log D_{\xi_1}) \hat{\otimes} \mathcal{E}^*(\overline{\mathcal{M}}((\xi_2)), \log D_{\xi_2}) = \mathsf{varG}((\Gamma)).$$

The same argument applies when the edge of Γ is a loop around a single vertex. We may now define, for any graph Γ with exactly one edge, a degree -1 map

$$c_\Gamma : \mathsf{varG}((g,n)) \to \mathsf{varG}((\Gamma)) \tag{5.90}$$

as the composition

$$\begin{aligned} \mathsf{varG}((g,n)) \;&=\; \mathcal{E}^*(\overline{\mathcal{M}}((g,n)), \log D((g,n))) \xrightarrow{\mathrm{Res}_\Gamma} \\ &\xrightarrow{\mathrm{Res}_\Gamma} \mathcal{E}^{*-1}(\overline{\mathcal{M}}((\Gamma)), \log D^2_\Gamma) \xrightarrow{Z_\Gamma} \mathsf{varG}((\Gamma)). \end{aligned}$$

Before going further, recall that the strong topological dual $V \mapsto V'$ defines an equivalence of the opposite category $\mathtt{NF}^{\mathrm{op}}$ with the category $\mathtt{DF}$ of dual Fréchet spaces, see again [53] or [42]. Thus $\mathsf{Grav}((g,n)) := \mathsf{varG}((g,n))'$ is a stable modular collection in the monoidal category $\mathtt{Chain(DF)}$ of chain complexes of dual Fréchet spaces. We believe that the meaning of the name varG is clear now – varG is Grav written from right to left.

Let us recall that the dualizing cocycle $\mathfrak{K}$ was, for $\Gamma \in \mathbf{\Gamma}((g,n))$, defined by

$$\mathfrak{K}(\Gamma) = \mathrm{Det}(\mathrm{Edg}(\Gamma)),$$

the top exterior power of $\mathrm{Span}_{\mathbf{k}}(\mathrm{Edg}(\Gamma))$ concentrated in degree $-\#(\mathrm{Edg}(\Gamma))$. For any $\Gamma \in \mathbf{\Gamma}((g,n))$ with exactly one edge the map (5.90) induces a degree 0 map

$$\alpha_\Gamma : \mathsf{Grav}((\Gamma)) \otimes \mathfrak{K}(\Gamma)^{-1} \to \mathsf{Grav}((g,n)) \tag{5.91}$$

by

$$\alpha_\Gamma(\omega \otimes \mathrm{Det}(e)) := 2\pi i\, c'_\Gamma(\omega),\ \omega \in \mathsf{varG}((\Gamma)),$$

where e is the unique edge of Γ. Iterating this map gives Grav a structure of a $\mathfrak{K}^{-1}$-modular operad, since the determinant in the definition of $\mathfrak{K}$ compensates for the anticommutativity of the diagram in Example 5.5.3.

The homology $\mathcal{G}rav((g,n))$ of $\mathsf{Grav}((g,n))$ form a modular $\mathfrak{K}^{-1}$-operad in the category of finite-dimensional vector spaces such that

$$\mathcal{G}rav((g,n)) \cong H_*(\mathcal{M}((g,n))).$$

An 'elementary' construction of this structure at the 'tree level' (for genus zero) was given by T. Kimura, J. Stasheff and A.A. Voronov in [58].

We owe the reader an explanation why we may assume that $\overline{\mathcal{M}}_{g,n}$ is smooth. The magic trick is that of an *algebraic stack*. For our purposes, an algebraic stack is a groupoid (a small category in which all morphisms are invertible) $\mathcal{G}$ such that both the set of objects $\mathrm{Ob}(\mathcal{G})$ and the set of morphisms $\mathrm{Mor}(\mathcal{G})$ are smooth manifolds. We also assume that the target and source maps $s,t : \mathrm{Mor}(\mathcal{G}) \to \mathrm{Ob}(\mathcal{G})$ are *étale*, which in our case means that they are local isomorphisms of smooth manifolds.

The definition of an algebraic stack is in fact more difficult. An intuitive idea of a stack is that of a scheme whose points admit nontrivial automorphisms. This is formalized by a certain category F fibered over the category of schemes, together with an *atlas*, which is, by definition, a scheme U together with a surjective étale (in an appropriate sense) map $U \to F$. From these data one construct a *presentation* of the stack, which is a category with objects U and morphisms the fibered product $U \times_F U$. Our groupoid $\mathcal{G}$ is then the set of complex points of the presentation. We recommend [129, Appendix] or the original paper [23] as an introduction to stacks.

A *coarse moduli space* $|\mathcal{G}|$ of an algebraic stack $\mathcal{G}$ is the quotient $\mathrm{Ob}(\mathcal{G})/\mathrm{Mor}(\mathcal{G})$. Notice that the coarse moduli space need not be smooth. A result of [23, Theorem 5.2] says that there exist an algebraic stack $\overline{\mathfrak{M}}_{g,n}$ such that the corresponding coarse moduli space is $\overline{\mathcal{M}}_{g,n}$.

The main trick is then to replace objects on $\overline{\mathcal{M}}_{g,n}$ with objects on $\mathrm{Ob}(\overline{\mathfrak{M}}_{g,n})$ that are invariant under the 'action' of $\mathrm{Mor}(\overline{\mathfrak{M}}_{g,n})$. For instance, a divisor D gives rise to a divisor in $\mathrm{Ob}(\overline{\mathfrak{M}}_{g,n})$ such that $s^{-1}(D) = t^{-1}(D)$. In the same manner, a sheaf $\mathcal{S}$ gives rise to a sheaf $\mathcal{S}$ on $\mathrm{Mor}(\overline{\mathfrak{M}}_{g,n})$ such that $s^*(\mathcal{S}) = t^*(\mathcal{S})$.

Example 5.5.4. An example one has to have in mind is a smooth manifold Y with an action of a finite group G. These objects define a stack $\mathcal{G}$ with $\mathrm{Ob}(\mathcal{G}) := Y$, $\mathrm{Mor}(\mathcal{G}) := G \times Y$ and the source and target maps given by

$$s(g \times y) := y,\ t(g \times y) := gy, \text{ for } g \in G,\ y \in Y.$$

The coarse moduli space $|\mathcal{G}|$ is the standard quotient Y/G and $\mathrm{Mod}(\mathcal{G})$-invariant objects are G-invariant objects on Y. Our trick then reflects the principle that, in characteristic zero, there is an one-to-one correspondence between objects on Y/G and G-invariant objects on Y.

As observed in [38], all constructions above can in fact be made for $\mathrm{Mor}(\overline{\mathfrak{M}}_{g,n})$-invariant objects in $\mathrm{Ob}(\overline{\mathfrak{M}}_{g,n})$. Let us summarize the results into the following theorem.

Theorem 5.5.5. *The collection* $\mathsf{Grav} = \{\mathsf{Grav}((g,n))\}_{g\geq 0, n\geq 1}$ *is a stable* $\mathfrak{K}^{-1}$*-modular operad in the monoidal category* $\mathtt{Chain(DF)}$ *of chain complexes of dual Fréchet spaces.*

The $\mathfrak{K}^{-1}$-modular operad Grav of Theorem 5.5.5 is called the *gravity operad*. We close this section with a statement describing the Feynman transform of Grav which we give without proof. Let us recall that the Feynman transform $\mathsf{F}_{\mathfrak{D}}$ introduced in **??** was a functor from the category of $\mathfrak{D}$-modular operads in the monoidal category of finite-dimensional vector spaces to the category of $\check{\mathfrak{D}}$-modular operads in the same category, where $\check{\mathfrak{D}} = \mathfrak{K} \otimes \mathfrak{D}^{-1}$. An obvious modification of $\mathsf{F}_{\mathfrak{D}}$ gives a functor $\mathsf{F}^{\mathrm{top}}_{\mathfrak{D}}$, the *topological* Feynman transform, from the category of $\mathfrak{D}$-modular operads in $\mathtt{Chain}(\mathtt{NF})$ to the category of $\check{\mathfrak{D}}$-modular operads in $\mathtt{Chain}(\mathtt{DF})$.

Recall that in **??** we introduced an invertible stable modular collection $\mathfrak{p}$ with $\mathfrak{p}((g,n)) = \downarrow^{6(g-1)-2n}\mathbf{k}$ and the property that $\mathfrak{D}_{\mathfrak{p}} = \mathfrak{K}^2$. From the last equation it easily follows that $\mathfrak{p}\mathsf{Grav}$ is a $\mathfrak{K}$-modular operad in $\mathtt{Chain}(\mathtt{DF})$, thus $\mathsf{F}^{\mathrm{top}}_{\mathfrak{K}}(\mathfrak{p}\mathsf{Grav})$ is a stable modular operad in $\mathtt{Chain}(\mathtt{NF})$.

There is another example of a modular $\mathtt{Chain}(\mathtt{NF})$-operad, namely the operad $\mathcal{C}_*(\overline{\mathcal{M}}) = \{\mathcal{C}_*(\overline{\mathcal{M}})((g,n))\}_{g\geq 0, n\geq 1}$, where $\mathcal{C}_*(-)$ denotes the complex of the deRham currents (see [26] for an introduction to deRham currents). The following proposition was proved in [38]:

Theorem 5.5.6. *The operads $\mathsf{F}^{\mathrm{top}}_{\mathfrak{K}}(\mathfrak{p}\mathsf{Grav})$ and $\mathcal{C}_*(\overline{\mathcal{M}})$ are weakly equivalent stable modular operads in the monoidal category* $\mathtt{Chain}(\mathtt{NF})$.

In the above theorem, weak equivalence means as usual the existence of a chain of operadic maps that are homology isomorphisms.

5.6. Application: Closed string field theory (general case).

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Let $\mathcal{H}$ be the Hilbert space of a combined conformal field theory of matter and ghosts and let $\mathcal{H}_{\mathrm{rel}} \subset \mathcal{H}$ be the subspace of elements annihilated by $b_0^- := b_0 - \overline{b}_0$ and $L_0^- := L_0 - \overline{L}_0$ (see, for example, [58, Section 4], or **??** of this book). Barton Zwiebach constructed in [136], for each 'genus' $g \geq 0$ and for each $n \geq 0$, multilinear 'string products'

$$\mathcal{H}_{\mathrm{rel}}^{\otimes n} \ni B_1 \otimes \cdots \otimes B_n \longmapsto [B_1, \ldots, B_n]_g \in \mathcal{H}_{\mathrm{rel}}.$$

Recall the basic properties of these products. If $\mathrm{gh}(-)$ denotes the ghost number, then [136, (4.8)]

$$\mathrm{gh}([B_1, \ldots, B_n]_g) = 3 - 2n + \sum_{i=1}^{n} \mathrm{gh}(B_i).$$

The string products are graded commutative [136, (4.4)]:

$$[B_1, \ldots, B_i, B_{i+1}, \ldots, B_n]_g = (-1)^{B_i B_{i+1}} [B_1, \ldots, B_{i+1}, B_i, \ldots, B_n]_g. \tag{5.92}$$

Here we used the notation

$$(-1)^{B_iB_{i+1}} := (-1)^{\mathrm{gh}(B_i)\mathrm{gh}(B_{i+1})}.$$

For $n = 0$ and $g \geq 0$, $[\,.\,]_g \in \mathcal{H}_{\mathrm{rel}}$ is just a constant, and the products are constructed in such a way that $[\,.\,]_0 = 0$ [136, (4.6)]. The linear operation $[B]_0 =: QB$ is identified with the BRST differential of the theory. These product satisfy, for all n, g, the *main identity* [136, (4.13)]:

$$0 = \sum \sigma(i_l, j_k)\,[B_{i_1}, \ldots, B_{i_l}, [B_{j_1}, \ldots, B_{j_k}]_{g_2}]_{g_1} \\ + \frac{1}{2}\sum_s (-1)^{\Phi_s}[\Phi_s, \Phi^s, B_1, \ldots, B_n]_{g-1}. \tag{5.93}$$

Here the first sum runs over all $g_1 + g_2 = g$, $k + l = n$, and all (k, l)-unshuffles $i_1 < \cdots < i_l$, $j_1 < \cdots < j_k$ of $\{1, \ldots, n\}$ (see **??** for the terminology). The sign $\sigma(i_l, j_k)$ is picked up by rearranging the sequence $(Q, B_1, \ldots, B_n)$ into the order $(B_{i_1}, \ldots, B_{i_l}, Q, B_{j_1}, \ldots, B_{j_k})$, where Q is the BRST differential of the theory, $\deg(Q) = 1$. In the second sum, $\{\Phi_s\}$ is a basis of $\mathcal{H}_{\mathrm{rel}}$ and $\{\Phi^s\}$ its dual basis in the sense that

$$(-1)^{\Phi_r}\langle \Phi_r, \Phi^s\rangle = \delta_r^s \text{ (Kronecker delta)}, \tag{5.94}$$

where $\langle -, - \rangle$ denotes the bilinear inner product on $\mathcal{H}$ [136, (2.44)]. Let us remark that, in the original formulation of [136], $\{\Phi_s\}$ was a basis of the whole $\mathcal{H}$, but the sum in (5.93) was restricted to $\mathcal{H}_{\mathrm{rel}}$. The product satisfies [136, (2.62)]:

$$\langle A, B\rangle = (-1)^{(A+1)(B+1)}\langle B, A\rangle \tag{5.95}$$

and it is nontrivial only for elements whose ghost numbers add up to five:

$$\text{if } \langle A, B\rangle \neq 0, \text{ then } \mathrm{gh}(A) + \mathrm{gh}(B) = 5. \tag{5.96}$$

The above two conditions in fact imply that $\langle A, B\rangle = \langle B, A\rangle$. Moreover, the product $\langle -, -\rangle$ is Q-invariant [136, 2.63]:

$$\langle QA, B\rangle = (-1)^A\langle A, QB\rangle. \tag{5.97}$$

Conditions (5.95) and (5.96) together with $\mathrm{gh}(\Phi^s) \equiv \mathrm{gh}(\Phi_s) + 1 \pmod 2$ imply that the element $\Phi := (-1)^{\Phi_s}\Phi_s \otimes \Phi^s \in \mathcal{H}_{\mathrm{rel}}^{\otimes 2}$ is symmetric in the sense that

$$(-1)^{\Phi_s}\Phi_s \otimes \Phi^s = (-1)^{\Phi_s}\Phi^s \otimes \Phi_s = -(-1)^{\Phi^s}\Phi^s \otimes \Phi_s. \tag{5.98}$$

We use, in the previous formula as well as at many places in the rest of the section, the Einstein convention of summing over repeated indices.

Let us prove (5.98). Since the form $\langle -, - \rangle$ is nondegenerate, to show that two elements of $\mathcal{H}_{\mathrm{rel}}^{\otimes 2}$ coincide, it is enough to prove that they are mapped by the map

$$\langle \Phi_r, - \rangle \langle \Phi_t, - \rangle : \mathcal{H}_{\mathrm{rel}} \otimes \mathcal{H}_{\mathrm{rel}} \to \mathbf{k}$$

to the same element, for any indices r, t. Thus, to prove the first equation of (5.98), it is enough to show that

$$(-1)^{\Phi_s} \langle \Phi_r, \Phi_s \rangle \langle \Phi_t, \Phi^s \rangle = (-1)^{\Phi_s} \langle \Phi_r, \Phi^s \rangle \langle \Phi_t, \Phi_s \rangle.$$

But, by (5.94), both sides of the above equation are equal to $\langle \Phi_r, \Phi_t \rangle$. This proves the first equation of (5.98). The second one follows from $\mathrm{gh}(\Phi^s) \equiv \mathrm{gh}(\Phi_s) + 1 \pmod 2$.

The last important property of string products is that the element

$$\Phi_s \otimes [\Phi^s, B_1, \ldots, B_{n-1}]_g \in \mathcal{H}_{\mathrm{rel}}^{\otimes 2} \tag{5.99}$$

is *antisymmetric*. This property is not explicitly stated in [136], but it follows from the fact that the string products are defined with the aid of the *multilinear string functions* [136, (7.72)]

$$\mathcal{H}_{\mathrm{rel}}^{\otimes (n+1)} \ni B_0 \otimes \cdots \otimes B_n \longmapsto \{B_0, \ldots, B_n\}_g \in \mathbb{C}.$$

The details can be found in [79].

Let us rewrite the axioms of string products into a more usual and convenient formalism so that it becomes manifest that they generalize strongly homotopy Lie algebras introduced in **??**. Recall that, for a graded vector space U, $\uparrow U$ (resp. $\downarrow U$) denotes the suspension (resp. the desuspension) of U, i.e. the graded vector space defined by $(\uparrow U)_p := U_{p-1}$ (resp. $(\downarrow U)_p := U_{p+1}$).

For a graded vector space U, let its *reflection* $\mathbf{r}(U)$ be the graded vector space defined by $\mathbf{r}(U)_p := U^{-p}$. There is an obvious natural map $\mathbf{r} : U \to \mathbf{r}(U)$. Observe that $\mathbf{r}^2 = \mathbb{1}$, $\mathbf{r} \circ \uparrow = \downarrow \circ\, \mathbf{r}$ and $\mathbf{r} \circ \downarrow = \uparrow \circ\, \mathbf{r}$.

Take now $V := \mathbf{r}(\downarrow \mathcal{H}_{\mathrm{rel}})$. Define, for each $g \geq 0$ and $n \geq 0$, multilinear maps $l_n^g : V^{\otimes n} \to V$ by

$$l_n^g(v_1, \ldots, v_n) := (-1)^{(n-1)v_1 + (n-2)v_2 + \cdots + v_{n-1}} \downarrow [\uparrow \mathbf{r}(v_1), \ldots, \uparrow \mathbf{r}(v_n)]_g, \text{ for } v_1, \ldots, v_n \in V^{\otimes n}.$$

Define also the bilinear form $B : V \otimes V \to \mathbb{C}$ by

$$B(u, v) := \langle \uparrow \mathbf{r}(u), \uparrow \mathbf{r}(v) \rangle. \tag{5.100}$$

Finally, let $h_s := (-1)^{\Phi_s} \mathbf{r}(\downarrow \Phi_s)$ and $h^s := \mathbf{r}(\downarrow \Phi^s)$, which means that

$$h_s \otimes h^s := (-1)^{\Phi_s} \mathbf{r}(\downarrow \Phi_s) \otimes \mathbf{r}(\downarrow \Phi^s).$$

The following definition abstracts the essentials of the above structure as can be verified by a technical, but absolutely straightforward, calculation.

Definition 5.6.1. *A* loop homotopy Lie algebra *is a triple $V = (V, B, \{l_n^g\})$ consisting of*

(i) a $\mathbb{Z}-$graded vector space V, $V_* = \bigoplus V_i$,

(ii) a graded symmetric nondegenerate bilinear degree $+3$ form $B : V \otimes V \to \mathbf{k}$, and

(iii) the set $\{l_n^g\}_{n,g\geq 0}$ of degree $n-2$ multilinear antisymmetric operations $l_n^g : V^{\otimes n} \to V$.

These data are supposed to satisfy the following two axioms.

(A1) For any $n, g \geq 0$ and $v_1, \ldots, v_n \in V$, the following 'main identity'

$$0 = \sum_{\substack{k+l=n+1\\ g_1+g_2=g\\ k\geq 1}} \sum_{\sigma\in \mathrm{unsh}(l,n-l)} \chi(\sigma)(-1)^{l(k-1)} l_k^{g_1}(l_l^{g_2}(v_{\sigma(1)}, \ldots, v_{\sigma(l)}), v_{\sigma(l+1)}, \ldots, v_{\sigma(n)}) \tag{5.101}$$
$$+\frac{1}{2}\sum_s (-1)^{h_s+n} l_{n+2}^{g-1}(h_s, h^s, v_1, \ldots, v_n)$$

holds. In the second sum, $\{h_s\}$ and $\{h^s\}$ are bases of the vector space V dual to each other in the sense that

$$B(h^s, h_t) = \delta_t^s. \tag{5.102}$$

(A2) The element

$$(-1)^{(n+1)h_s} h_s \otimes l_n^g(h^s, v_2, \ldots, v_n) \in V \otimes V \tag{5.103}$$

is symmetric, for all $g \geq 0, n \geq 0$, and $v_2, \ldots, v_n \in V$.

We are going to show how 'abstract' loop homotopy Lie algebras are related to a certain construction over the operad *Com* for commutative algebras. All algebraic objects in the rest of this section are defined over a field $\mathbf{k}$ of characteristic zero.

Remark 5.6.2. To give a reasonable meaning to the 'basis $\{h_s\}$ of V,' we must suppose either that V is finite dimensional, or that it has a suitable topology, as in the case of string products. We will always tacitly assume that assumptions of this form have been made. In the 'main identity' for $g = 0$ we put, by definition, $l_n^{-1} = 0$.

Because $\deg(h_s) + \deg(h^s) = -3$, we have that $\deg(h_s)\deg(h^s)$ is even. The graded symmetry of B then implies that, in addition to (5.102), also $B(h_s, h^t) = \delta_s^t$. The element $h = h_s \otimes h^s$ is easily seen to be symmetric, $h_s \otimes h^s = (-1)^{h_s h^s} h^s \otimes h_s = h^s \otimes h_s$.

For $n = 0$ axiom (5.93) gives

$$0 = \sum_{g_1+g_2=g} l_1^{g_1}(l_0^{g_2}(.)) + \frac{1}{2}\sum_s (-1)^{h_s} l_2^{g-1}(h_s, h^s),$$

while for $n = 1$ it gives

$$0 = \sum_{g_1+g_2=g} (l_1^{g_1}(l_1^{g_2}(v)) + l_2^{g_1}(l_0^{g_2}(.), v)) - \frac{1}{2}\sum_s (-1)^{h_s} l_3^{g-1}(h_s, h^s, v), \text{ for all } v \in V. \tag{5.104}$$

From this moment on, we will assume that $l_0^g = 0$, for all $g \geq 0$, that is, the theory has 'no constants.' This assumption is not really necessary, but it will considerably simplify our exposition.

Exercise 5.6.3. Let us denote $\partial := l_1^0$. Equation (5.104) implies that $\partial^2 = 0$ (recall our assumption $l_0^g = 0$!). Thus ∂ is a degree -1 differential on the space V. The symmetry of $h_s \otimes \partial(\tilde{h}_s)$ (axiom (A2) with $n = 1$ and $g = 1$) is equivalent to the d-invariance of the form B, that is, for $u, v \in V$, $B(\partial u, v) + (-1)^u B(u, \partial v) = 0$.

The tree level. Let us discuss the 'tree level' ($g = 0$) specialization of the above structure, when the only nontrivial l_n^g's are $l_n := l_n^0$, $n \geq 1$. The main identity (5.101) for $g = 0$ reduces to

$$0 = \sum_{k+l=n+1} \sum_{\sigma \in \mathrm{unsh}(l,n-l)} \chi(\sigma)(-1)^{l(k-1)} l_k(l_l(v_{\sigma(1)}, \ldots, v_{\sigma(l)}), v_{\sigma(l+1)}, \ldots, v_{\sigma(n)}). \tag{5.105}$$

We immediately recognize (5.105) as the defining equation for strongly homotopy Lie algebras **??**.

For $g = 1$, the main identity gives (after forgetting the overall factor $\frac{(-1)^n}{2}$)

$$0 = \sum_s (-1)^{h_s} l_{n+2}(h_s, h^s, v_1, \ldots, v_n). \tag{5.106}$$

Axiom (A2) says that the elements

$$(-1)^{(n+1)h_s} h_s \otimes l_n(h^s, v_1, \ldots, v_n) \tag{5.107}$$

are symmetric. Thus the tree level loop homotopy Lie algebra is a strongly homotopy Lie algebra $(V, \{l_n\})$ which acquires an additional structure given by a bilinear form B such that the element $h = h_s \otimes h^s$, uniquely determined by B, satisfies (5.106).

We see that the 'tree-level' specialization is a richer structure than just a strongly homotopy Lie algebra as it is usually understood. A proper name for such a structure would be a *cyclic* strongly homotopy Lie algebra, compare [63, 96].

5.6.1. Operadic interpretation.

In this subsection we show that loop homotopy algebras (without 'constants,' see Remark 5.6.2) are algebras over a certain modular operad, constructed from the cyclic operad $\mathcal{C}om$ for commutative algebras (Theorem 5.6.9 and Theorem 5.6.10). The constants can be easily incorporated by applying the same construction to the operad $\mathcal{UC}om$ for *unital* commutative associative algebras.

Let us consider the following harmless modification of Example **??**.

Example 5.6.4. Let $W = (W, H)$ be a graded vector space with a nondegenerate degree -1 symmetric bilinear form H. Define the modular collection $End_W^{\mathfrak{K}}$ by

$$End_W^{\mathfrak{K}}((g,S)) := W^{\otimes S},$$

for $g \geq 0$ and a finite set S. For $\Gamma \in \mathbf{\Gamma}((g,S))$, the $\mathfrak{K}$-modular 'composition map'

$$\alpha_\Gamma^{\mathfrak{K}} : End_W^{\mathfrak{K}}((\Gamma)) \otimes \mathfrak{K}(\Gamma) \to End_W^{\mathfrak{K}}((g,S))$$

is defined as in (**??**). This means that we choose labels s_e, t_e such that $e = \{s_e, t_e\}$ for each edge $e \in \mathrm{Edg}(\Gamma)$ and define $\alpha_\Gamma^{\mathfrak{K}}$ to be the composition:

$$\begin{aligned} End_W^{\mathfrak{K}}((\Gamma)) \otimes \mathfrak{K}(\Gamma) &\cong W^{\otimes \mathrm{Flag}(\Gamma)} \otimes \mathrm{Det}(\mathrm{Edg}(\Gamma)) \cong W^{\otimes S} \otimes \bigotimes_{e \in \mathrm{Edg}(\Gamma)} \Big(W^{\otimes \{s_e, t_e\}} \otimes \mathrm{Span}(\downarrow e) \Big) \qquad (5.108) \\ &\cong W^{\otimes S} \otimes \bigotimes_{e \in \mathrm{Edg}(\Gamma)} \Big(W_{s_e} \otimes W_{t_e} \otimes \mathrm{Span}(\downarrow e) \Big) \\ &\xrightarrow{1\!\!1 \otimes \bigotimes_e H_e} W^{\otimes S} \otimes \mathbf{k}^{\otimes \mathrm{Edg}(\Gamma)} \cong End_W^{\mathfrak{K}}((g,S)), \end{aligned}$$

where H_e is the map that sends $u \otimes v \otimes \downarrow e \in W_{s_e} \otimes W_{t_e} \otimes \mathrm{Span}(\downarrow e)$ to $H(u,v) \in \mathbf{k}$. As in Example **??**, the symmetry of H implies that the the definition of $\alpha_\Gamma^{\mathfrak{K}}$ does not depend on the choice of labels. It is easy to verify that $\{\alpha_\Gamma^{\mathfrak{K}} |\ \Gamma \in \mathbf{\Gamma}((g,S))\}$ induces on $End_W^{\mathfrak{K}}$ the structure of a $\mathfrak{K}$-modular operad.

By definition, a loop homotopy Lie algebra is a structure that lives on a graded vector space $V = (V, B)$ with a nondegenerate degree $+3$ bilinear symmetric form B. Then $W = (W, H)$ with $W := \uparrow^2 V$ and the form H defined by $H(u,v) := B(\downarrow^2 u, \downarrow^2 v)$, $u, v \in W$, provide the data as in Example 5.6.4 and we may consider the modular $\mathfrak{K}$-operad $End_{\uparrow^2 V}^{\mathfrak{K}}$.

Let us consider the functor $\Box : \mathtt{MOp} \to \mathtt{COp}$ from the category of modular operads to the category of cyclic operads given, for any finite set S, by $\Box(\mathcal{A})((S)) := \mathcal{A}((0,S))$. We are going to describe a left adjoint $\mathrm{Mod} : \mathtt{COp} \to \mathtt{MOp}$ to $\Box$.

Let $\underline{\mathbf{\Gamma}}((g,S))$ be the subcategory of the category $\mathbf{\Gamma}((g,S))$ whose objects are graphs $\Gamma \in \mathbf{\Gamma}((g,S))$ such that $g(v) = 0$ for each vertex $v \in \mathrm{Vert}(\Gamma)$. A morphism $f : \Gamma_0 \to \Gamma_1 \in \mathbf{\Gamma}((g,S))$ is a morphism in $\underline{\mathbf{\Gamma}}((g,S))$ if and only if f has a factorization (**??**) such that each π_{e_i} is a contraction along an edge joining two *distinct* vertices. Another, equivalent way to characterize morphisms $f : \Gamma_0 \to \Gamma_1$ of $\underline{\mathbf{\Gamma}}((g,S))$ is that the graph $f^{-1}(v)$ is a tree, for any $v \in \mathrm{Vert}(\Gamma_1)$.

For a cyclic operad $\mathcal{P}$ and a graph $\gamma \in \underline{\mathbf{\Gamma}}((g,S))$, consider the graded vector space

$$\mathcal{P}((\Gamma)) = \bigotimes_{v \in \mathrm{Vert}(\Gamma)} \mathcal{P}((\mathrm{Leg}(v))).$$

For each $f : \Gamma_0 \to \Gamma_1 \in \underline{\mathbf{\Gamma}}((g, S))$ define a morphism $\mathcal{P}((f)) : \mathcal{P}((\Gamma_0)) \to \mathcal{P}((\Gamma_1))$ by

$$\begin{aligned} \mathcal{P}((\Gamma_0)) &= \bigotimes_{u \in \mathrm{Vert}(\Gamma_0)} \mathcal{P}((\mathrm{Leg}(u))) \cong \bigotimes_{v \in \mathrm{Vert}(\Gamma_1)} \mathcal{P}((f^{-1}(v))) \qquad (5.109) \\ &\xrightarrow{\bigotimes_v p_{f^{-1}(v)}} \bigotimes_{v \in \mathrm{Vert}(\Gamma_1)} \mathcal{P}((\mathrm{Leg}(v))) = \mathcal{P}((\Gamma_1)). \end{aligned}$$

where $p_f^{-1}(v)$ is the $\mathcal{P}$-composition along the graph $f^{-1}(v)$. This composition exists because $f^{-1}(v)$ is a tree, by assumption. The definition of the map $\mathcal{P}((f))$ is formally the same as that of (**??**), except the graphs Γ_1 and Γ_2 need not be trees. We have the following analog of Proposition 5.3.8.

Proposition 5.6.5. *The correspondence $\Gamma \mapsto \mathcal{P}((\Gamma))$, $f \mapsto p(f)$ defines, for each $g \geq 0$ and each finite set S, a functor from the category $\underline{\mathbf{\Gamma}}((g, S))$ to the category of graded vector spaces.*

By the above proposition, it makes sense to put, for any $\mathcal{P} \in \mathtt{COp}$,

$$\mathrm{Mod}(\mathcal{P})((g, S)) := \operatorname*{colim}_{\Gamma \in \underline{\mathbf{\Gamma}}((g,S))} \mathcal{P}((\Gamma)).$$

Observe the analogy to the definition of Γ_K in **??** where the colimit is taken over the compositions that are already defined for a K-collection, whereas here the colimit is over the operadic 'compositions' already defined for a cyclic operad.

The modular operad structure on the modular collection $\mathrm{Mod}(\mathcal{P})$ can be defined as follows. Consider the category $\underline{\mathbf{\Gamma}}_1((g, S))$ whose objects are graph morphisms $\Gamma_0 \xrightarrow{f} \Gamma_1$, where $\Gamma_0 \in \underline{\mathbf{\Gamma}}((g, S))$, $\Gamma_1 \in \mathbf{\Gamma}((g, S))$, and morphisms are commutative diagrams

$$\begin{tikzcd} \Gamma_0 \arrow[r, "f"] \arrow[d, "u_0"'] & \Gamma_1 \arrow[d, "u_1"] \\ \Gamma_0' \arrow[r, "f'"] & \Gamma_1' \end{tikzcd}$$

where u_0 is a morphism in $\underline{\mathbf{\Gamma}}((g, S))$ and u_1 is an isomorphism. We may show, as in **??**, that

$$\mathbb{M}(\mathrm{Mod}(\mathcal{P})) \cong \operatorname*{colim}_{[\Gamma_0 \xrightarrow{f} \Gamma_1] \in \underline{\mathbf{\Gamma}}_1((g,S))} \mathcal{P}((\Gamma_0)).$$

The functor $\underline{\mathbf{\Gamma}}_1((g, S)) \to \underline{\mathbf{\Gamma}}((g, S))$ that sends $[\Gamma_0 \xrightarrow{f} \Gamma_1]$ to Γ_0 defines a map $\mathbb{M}(\mathrm{Mod}(\mathcal{P})) \to \mathrm{Mod}(\mathcal{P})$. It follows from the same arguments as in (**??**) that this map induces a modular operad structure on $\mathrm{Mod}(\mathcal{P})$.

Definition 5.6.6. *The modular operad* $\mathrm{Mod}(\mathcal{P})$ *is called the* modular operadic completion *of the cyclic operad* $\mathcal{P}$.

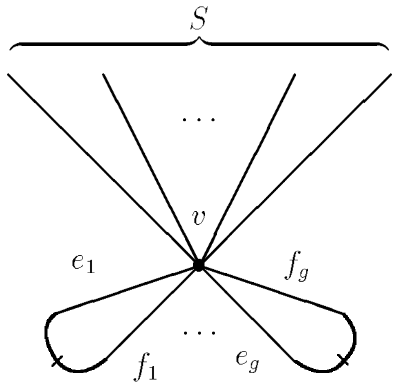


Figure 5.4: The graph $\Upsilon_{(\!(g,S)\!)}$.

Proposition 5.6.7. *The functor* Mod : $\mathtt{COp} \to \mathtt{MOp}$ *is a left adjoint to the forgetful functor* $\Box : \mathtt{MOp} \to \mathtt{COp}$.

Since we will not need the proposition, we omit its proof. There is no easy way to express $\mathrm{Mod}(\mathcal{P})$ in terms of $\mathcal{P}$. The category $\underline{\mathbf{\Gamma}}((g,S))$ has, for $g > 0$, no terminal object, though there exist an important distinguished graph $\Upsilon_{(\!(g,S)\!)}$ described as follows.

The graph $\Upsilon_{(\!(g,S)\!)}$ has one vertex v of genus 0, the set of legs $\mathrm{Leg}(\Upsilon_{(\!(g,S)\!)}) := S$, the set of internal flags $\mathrm{IFlag}(\Upsilon_{(\!(g,S)\!)}) := \{e_1, f_1, \dots, e_g, f_g\}$ and the involution σ acting on $\mathrm{IFlag}(\Upsilon_{(\!(g,S)\!)})$ by $\sigma(e_i) = f_i$, for $1 \leq i \leq g$. The geometric realization of the graph $\Upsilon_{(\!(g,S)\!)}$ is depicted in Figure 5.4.

The most important property of the graph $\Upsilon_{(\!(g,S)\!)}$ is that

$$\text{for each } \Gamma \in \underline{\mathbf{\Gamma}}((g,S)) \text{ there exists a map } f : \Gamma \to \Upsilon_{(\!(g,S)\!)} \in \underline{\mathbf{\Gamma}}((g,S)), \tag{5.110}$$

but the map f need not be unique. In fact, the automorphism group of the graph $\Upsilon_{(\!(g,S)\!)}$ is quite large, namely $\mathrm{Aut}(\Upsilon_{(\!(g,S)\!)}) \cong \Sigma_g \rtimes (\mathbb{Z}_2)^{\times g}$, the semidirect product of the symmetric group Σ_g and the g-fold cartesian power of $\mathbb{Z}_2$, with Σ_g acting on $(\mathbb{Z}_2)^{\times g}$ by permuting the factors. The graph $\Upsilon_{(\!(g,n)\!)}$ can still, however, be used to define a map $\rho : \mathcal{P}((n+2g)) \to \mathrm{Mod}(\mathcal{P})((g,n))$, for any $g \geq 0$ and $n \geq 1$, as follows.

The order $(0, 1, \dots, n, e_1, f_1, \dots, e_g, f_g)$ of $\mathrm{Flag}(\Upsilon_{(\!(g,n)\!)})$, defines an isomorphism $P((n+2g)) \cong \mathcal{P}((\Upsilon_{(\!(g,n)\!)}))$, which then composes with the projection to yeld the canonical map

$$\rho : P((n+2g)) \xrightarrow{\cong} \mathcal{P}((\Upsilon_{(\!(g,n)\!)})) \longrightarrow \operatorname*{colim}_{\Gamma \in \underline{\mathbf{\Gamma}}((g,S))} \mathcal{P}((\Gamma)) = \mathrm{Mod}(\mathcal{P})((g,n)). \tag{5.111}$$

The map ρ is always an epimorphism, as easily follows from (5.110) but, in general, it need not be an isomorphism (the map ρ cannot distinguish between elements which differ only by the action of $\mathrm{Aut}(\Upsilon_{(\!(g,n)\!)})$).

Example 5.6.8. Since $Com((n)) = \mathbf{k}$ for $n \geq 1$, the vector space $Com((\Gamma))$ is canonically isomorphic to $\mathbf{k}$ and, for each map $f : \Gamma_0 \to \Gamma_1 \in \underline{\mathbf{\Gamma}}((g,S))$, the induced map $Com((f)) : Com((\Gamma_0)) \to Com((\Gamma_1))$ is, under this identification, the identity,

$$Com((f)) : Com((\Gamma_0)) \cong \mathbf{k} \xrightarrow{1\!\!1} \mathbf{k} \cong Com((\Gamma_1)).$$

The map $\rho : Com((n+2g)) \to \mathrm{Mod}(Com)((g,n))$ of (5.111) is in this case easily seen to be an isomorphism, therefore

$$\mathrm{Mod}(Com)((g,n)) \cong \mathbf{k}, \text{ for each } g \geq 0,\ n \geq 1, \tag{5.112}$$

with the trivial action of the symmetric group Σ_{n+1}. To determine the modular operad structure, observe that, for each $\Gamma \in \mathbf{\Gamma}((g,S))$, also $\mathrm{Mod}(Com)((\Gamma))$ is canonically isomorphic to $\mathbf{k}$. The structure map is, under this identification, the isomorphism

$$\alpha_\Gamma : \mathrm{Mod}(Com)((\Gamma)) \cong \mathbf{k} \xrightarrow{1\!\!1} \mathbf{k} \cong \mathrm{Mod}(Com)((g,S)).$$

Let us recall that in **??** we described the Feynman transform of a modular operad as an analog of the (co)bar construction of an ordinary operad. We are going to describe explicitly the Feynman transform $\mathsf{F} = \mathsf{F}(\mathrm{Mod}(Com)) = (\mathsf{F}(\mathrm{Mod}(Com)), \partial_{\mathsf{F}})$ of the modular operad $\mathrm{Mod}(Com)$. Ignoring the differential, it is, by definition, the free $\mathfrak{K}$-modular operad $\mathbb{M}_{\mathfrak{K}}((\mathrm{Mod}(Com))^*)$ on the linear dual of the modular collection $\mathrm{Mod}(Com)$.

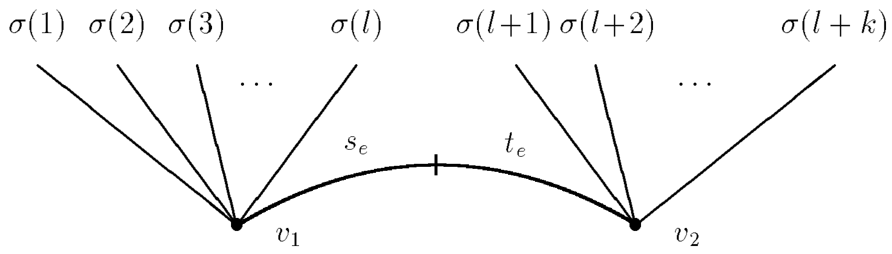
By (5.112), $\mathrm{Mod}(Com)((g,n)) \cong \mathbf{k}$, for all $g \geq 0$, $n \geq 1$. Let $\omega_n^g \in (\mathrm{Mod}(Com))^*((g,n))$ be the dual of $1 \in \mathbf{k} \cong \mathrm{Mod}(Com)((g,n))$. Therefore, as a modular collection,

$$\mathsf{F}(\mathrm{Mod}(Com)) = \mathbb{M}_{\mathfrak{K}}\Big(\{\omega_n^g |\ g \geq 0,\ n \geq 1\}\Big). \tag{5.113}$$

Since the action of the symmetric group Σ_{n+1} on $\mathrm{Mod}(Com)^*((g,n))$ is trivial, the vector space $\mathrm{Mod}(Com)^*((g,S))$ is *canonically* isomorphic to $\mathrm{Mod}(Com)^*((g,n))$ for *any* finite set S with $\mathrm{card}(S) = n+1$, thus ω_n^g determines the element (denoted by the same symbol) $\omega_n^g \in \mathrm{Mod}(Com)^*((g,S))$. Therefore each graph $\Gamma \in \mathbf{\Gamma}((g,S))$ determines the *canonical element*

$$\gamma := \bigotimes_{v \in \mathrm{Vert}(\Gamma)} \omega^{g(v)}_{|\mathrm{Flag}(v)|-1} \in \bigotimes_{v \in \mathrm{Vert}(\Gamma)} \mathrm{Mod}(Com)^*((g(v), \mathrm{Flag}(v))) = \mathrm{Mod}(Com)^*((\Gamma)).$$

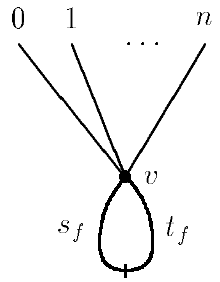
For $g_1, g_2 \geq 0$ with $g_1 + g_2 = g$, $k, l \geq 1$ with $k + l = n+1$ and $\sigma \in \mathrm{unsh}(k,l)$, let $\Phi^{g_1,g_2}_{k,l}(\sigma) \in \mathbf{\Gamma}((g,n))$ be the graph



with $g(v_1) = g_1$ and $g(v_2) = g_2$. Let

$$\phi^{g_1,g_2}_{k,l}(\sigma) \in \mathrm{Mod}(\mathcal{C}om)^*((\Phi^{g_1,g_2}_{k,l}(\sigma)))$$

be the canonical element. In a similar manner, let $\Psi^g_n \in \mathbf{\Gamma}((g,n))$ be the graph



where $g(v) = g-1$ and let $\psi^g_n \in \mathrm{Mod}(\mathcal{C}om)^*((\Psi^g_n))$ be the corresponding canonical element. It is immediate from the definition (**??**) of the differential ∂_{F} that

$$\partial_{\mathsf{F}}(\omega^g_n) = \sum_{\substack{k+l=n+1,\ k,l\geq 1\\ g_1+g_2=g,\ \sigma\in \mathrm{unsh}(k,l)}} \phi^{g_1,g_2}_{k,l}(\sigma)\otimes \uparrow e + \psi^g_n \otimes \uparrow f, \tag{5.114}$$

where $e := \{s_e, t_e\} \in \mathrm{Edg}(\Phi^{g_1,g_2}_{k,l}(\sigma))$ and $f := \{s_f, t_f\} \in \mathrm{Edg}(\Psi^g_n)$. The obvious symmetry

$$\Phi^{g_1,g_2}_{k,l}(\sigma) \longleftrightarrow \Phi^{g_2,g_1}_{l,k}(\sigma'),$$

where σ' is, for $\sigma \in \mathrm{unsh}(k,l)$, the 'opposite unshuffle,'

$$\sigma' := (\sigma(k+1),\ldots,\sigma(k+l),\sigma(1),\ldots,\sigma(k)),$$

enables one to rewrite (5.114) as

$$\partial_{\mathsf{F}}(\omega^g_n) = \sum_{\substack{k+l=n+1,\ k,l\geq 1\\ g_1+g_2=g,\ \sigma\in \mathrm{unsh}_0(k,l)}} 2\phi^{g_1,g_2}_{k,l}(\sigma)\otimes \uparrow e + \psi^g_n \otimes \uparrow f, \tag{5.115}$$

where $\mathrm{unsh}_0(k,l) := \{\sigma \in \mathrm{unsh}(k,l) |\ \sigma(1) = 1\}$.

Let us recall the following notation. For $u_1\otimes\cdots\otimes u_r \in V^{\otimes r}$ and $v_1\otimes\cdots\otimes v_s \in V^{\otimes s}$, let $\mathrm{Sh}(u_1\otimes\cdots\otimes u_r|v_1\otimes\cdots\otimes v_s)$ denote the shuffle product of $u_1\otimes\cdots\otimes u_k$ with $v_1\otimes\cdots\otimes v_l$ in $TV = \bigoplus_{n\geq 0} V^{\otimes n}$, i.e. the sum of all shuffles of $u_1\otimes\cdots\otimes u_r$ and $v_1\otimes\cdots\otimes v_s$. For example

$$\begin{aligned}
\mathrm{Sh}(u_1{\otimes}u_2|v_1{\otimes}v_2{\otimes}v_3) = {} & u_1{\otimes}u_2{\otimes}v_1{\otimes}v_2{\otimes}v_3 + (-1)^{u_2v_1}u_1{\otimes}v_1{\otimes}u_2{\otimes}v_2{\otimes}v_3 + \\
& +(-1)^{u_2(v_1+v_2)}u_1{\otimes}v_1{\otimes}v_2{\otimes}u_2{\otimes}v_3 + (-1)^{u_2(v_1+v_2+v_3)}u_1{\otimes}v_1{\otimes}v_2{\otimes}v_3{\otimes}u_2 + \\
& +(-1)^{v_1(u_1+u_2)}v_1{\otimes}u_1{\otimes}u_2{\otimes}v_2{\otimes}v_3 + +(-1)^{v_1(u_1+u_2)+u_2v_2}v_1{\otimes}u_1{\otimes}v_2{\otimes}u_2{\otimes}v_3 + \\
& +(-1)^{v_1(u_1+u_2)+u_2(v_2+v_3)}v_1{\otimes}u_1{\otimes}v_2{\otimes}v_3{\otimes}u_2 + (-1)^{(u_1+u_2)(v_1+v_2)}v_1{\otimes}v_2{\otimes}u_1{\otimes}u_2{\otimes}v_3 + \\
& +(-1)^{(u_1+u_2)(v_1+v_2)+u_2v_3}v_1{\otimes}v_2{\otimes}u_1{\otimes}v_3{\otimes}u_2 + (-1)^{(u_1+u_2)(v_1+v_2+v_3)}v_1{\otimes}v_2{\otimes}v_3{\otimes}u_1{\otimes}u_2,
\end{aligned}$$

see also ??. A general element $x \in V^{\otimes k}$ is of the form

$$x = \sum x_{i_1} \otimes \cdots \otimes x_{i_k} \text{ where } x_{i_1}, \ldots, x_{i_k} \in V,$$

where the summation runs over an appropriate set of indices. We usually omit the summation symbol and write simply

$$x = x_{i_1} \otimes \cdots \otimes x_{i_k}.$$

Let $V = (V, B)$ be a graded vector space with a nondegenerate degree $+3$ symmetric bilinear form and let $\{h_s\}$, $\{h^s\}$ be the bases of V as in Definition 5.6.1.

Theorem 5.6.9. *A $\mathfrak{K}$-modular $\mathsf{F}(\mathrm{Mod}(Com))$-algebra structure on $\uparrow^2 V = (\uparrow^2 V, B)$ is given by a sequence $\{\xi^g_n |\ g \geq 0, n \geq 1\}$, where $\xi^g_n = x^g_{i_0} \otimes \cdots \otimes x^g_{i_n} \in V^{\otimes n+1}$ is a degree $-2(n+1)$ symmetric element of $V^{\otimes n+1}$ such that, for each $n \geq 1$ and $g \geq 0$, the following 'main equation' is satisfied:*

$$0 = \sum_{\substack{k+l=n+1,\ k,l\geq 1\\ g_1+g_2=g}} (-1)^{x^{g_1}_{i_k}} B(x^{g_1}_{i_k}, x^{g_2}_{j_0}) x^{g_1}_{i_0} \otimes \mathrm{Sh}(x^{g_1}_{i_1} \otimes \cdots \otimes x^{g_1}_{i_{k-1}} | x^{g_2}_{j_1} \otimes \cdots \otimes x^{g_2}_{j_l}) \tag{5.116}$$
$$- \frac{1}{2} B(x^{g-1}_{i_{n+1}}, x^{g-1}_{i_{n+2}}) x^{g-1}_{i_0} \otimes \cdots \otimes x^{g-1}_{i_n}.$$

In the above identity for $g = 0$ we put, by definition, $\xi^{g-1}_{n+2} = x^{g-1}_{i_0} \otimes \cdots \otimes x^{g-1}_{i_{n+1}} := 0$.

Proof. An $\mathsf{F}(\mathrm{Mod}(Com))$-algebra structure on $(\uparrow^2 V, B)$ is an $\mathfrak{K}$-modular differential operad map

$$a : (\mathsf{F}(\mathrm{Mod}(Com)), \partial_{\mathsf{F}}) \to (End^{\mathfrak{K}}_{\uparrow^2 V}, \partial = 0).$$

Description (5.113) shows that such a map a is given by its values $\xi^g_n := a(\omega^g_n)$ on the generators. Moreover, the map a commutes with the differentials, so the equation $a(\partial_{\mathsf{F}}(\omega^g_n)) = 0$ must be satisfied. This is, by (5.115), the same as

$$a\Big(\sum_{\substack{k+l=n+1,\ k,l\geq 1\\ g_1+g_2=g,\ \sigma\in \mathrm{unsh}_0(k,l)}} 2\phi^{g_1,g_2}_{k,l}(\sigma) \otimes \uparrow e + \psi^g_n \otimes \uparrow f \Big) = 0. \tag{5.117}$$

The 'main identity' (5.116) is then (5.117) expressed in terms of the structure operations of the endomorphism operad $End^{\mathfrak{K}}_{\uparrow^2 V}$. ■

Let us introduce our basic tool which converts the structures of Theorem 5.6.9 into loop homotopy algebras, the isomorphism $\Xi : V^{\otimes n+1} \to Hom V^{\otimes n} V$. The non-degenerated bilinear form B identifies the space V with its dual, $V \cong V^{\#}$, thus $V^{\otimes n+1} \cong V^{\#^{\otimes n}} \otimes V$, while, as always, $V^{\#^{\otimes n}} \otimes V \cong Hom V^{\otimes n} V$. Thus the existence of Ξ is not a surprise. The subtle issue is a proper choice of signs, which is provided by the formula

$$\Xi(x_0 \otimes \cdots \otimes x_n)(v_1, \ldots, v_n) := \tag{5.118}$$
$$:= (-1)^{nx_0 + (n-1)x_1 + \cdots + x_{n-1}} x_0 B(x_1, v_1) B(x_2, v_2) \cdots B(x_n, v_n),$$

for $x_0 \otimes \cdots \otimes x_n \in V^{\otimes n+1}$ and $v_1, \ldots, v_n \in V$. The map Ξ is clearly a degree $3n$ isomorphism of $V^{\otimes n+1}$ and $Hom V^{\otimes n} V$. Observe that $\Xi(x_0 \otimes \cdots \otimes x_n)(v_1, \ldots, v_n) \neq 0$ only if $\deg(x_i) = \deg(v_i) + 1 \pmod 2$, $1 \leq i \leq n$.

Theorem 5.6.10. *Let us define, for $g \geq 0$, $n \geq 1$ and $v_1, \ldots, v_n \in V$,*

$$l_n^g(v_1, \ldots, v_n) := (-1)^{\frac{n(n+1)}{2} + n(v_1 + \cdots + v_n)} \Xi(\xi_n^g)(v_1, \ldots, v_n).$$

Then $\{\xi_n^g\} \mapsto \{l_n^g\}$ is a one-to-one correspondence between structures of Theorem 5.6.9 and loop homotopy algebras in the sense of Definition 5.6.1.

The proof of Theorem 5.6.10, based on the properties of the map Ξ, is mostly a tedious exercise in getting the signs right. It is given in the following appendix.

5.6.2. Appendix.

The first step in the proof of Theorem 5.6.10 is the following lemma, which describes the behavior of Ξ under the symmetric group action.

Lemma 5.6.11. *Let $x_0 \otimes \cdots \otimes x_n \in V^{\otimes n+1}$ and $v_1 \otimes \cdots \otimes v_n \in V^{\otimes n}$. Then, for any $\sigma \in \Sigma_n$,*

$$\begin{aligned} &\epsilon(\sigma) \Xi(x_0 \otimes x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)})(v_1, \ldots, v_n) = \\ &= \chi(\sigma) \Xi(x_0 \otimes x_1 \otimes \cdots \otimes x_n)(v_{\sigma^{-1}(1)}, \ldots, v_{\sigma^{-1}(n)}), \end{aligned} \tag{5.119}$$

where $\epsilon(\sigma) = \epsilon(\sigma; x_1, \ldots, x_n)$ is the Koszul sign (??) of the permutation σ and $\chi(\sigma) = \chi(\sigma; v_1, \ldots, v_n) := \mathrm{sgn}(\sigma) \epsilon(\sigma; v_1, \ldots, v_n)$. Moreover,

$$\begin{aligned} &\Xi(x_1 \otimes x_0 \otimes x_2 \otimes \cdots \otimes x_n)(v_1, \ldots, v_n) = \\ &= (-1)^{x_0 + x_1} h_s \otimes B\Big(\Xi(x_0 \otimes x_1 \otimes \cdots \otimes x_n)(h^s, v_2, \ldots, v_n), v_1\Big). \end{aligned} \tag{5.120}$$

Proof. Let us prove (5.119). Since the symmetric group is generated by transpositions, it is clearly enough to show that

$$\begin{aligned} &(-1)^{x_i x_{i+1}} \Xi(x_0 \otimes x_1 \otimes \cdots \otimes x_{i+1} \otimes x_i \otimes \cdots \otimes x_n)(v_1, \ldots, v_n) = \\ &= -(-1)^{v_i v_{i+1}} \Xi(x_0 \otimes x_1 \otimes \cdots \otimes x_n)(v_1, \ldots, v_{i+1}, v_i, \ldots, v_n), \end{aligned}$$

for all $1 \leq i < n$. By (5.118), the left hand side of the above equation is

$$(-1)^{x_i x_{i+1} + n x_0 + \cdots + (n-i) x_{i+1} + (n-i-1) x_i + \cdots + x_{n-1}} x_0 B(x_1, v_1) \cdots B(x_{i+1}, v_i) B(x_i, v_{i+1}) \cdots B(x_n, v_n)$$

while the right hand side equals to

$$-(-1)^{v_i v_{i+1} + n x_0 + \cdots + (n-i) x_i + (n-i-1) x_{i+1} + \cdots + x_{n-1}} x_0 B(x_1, v_1) \cdots B(x_i, v_{i+1}) B(x_{i+1}, v_i) \cdots B(x_n, v_n).$$

Modulo signs, the above two expressions are the same, while their signs differ by

$$-(-1)^{v_iv_{i+1}+x_ix_{i+1}+x_i+x_{i+1}}$$

which is 1, because $v_i = x_{i+1} + 1 \pmod 2$ and $v_{i+1} = x_i + 1 \pmod 2$, if we assume (5.119) to be nontrivial.

Equation (5.120) can be proved in the same straightforward manner. ■

Corollary 5.6.12. *The map Ξ induces an isomorphism between the space of symmetric elements $x = x_{i_0} \otimes \cdots \otimes x_{i_n} \in V^{\otimes n+1}$ and the space of graded antisymmetric maps $k : V^{\otimes n} \to V$ with the property that the element*

$$(-1)^{h^s} h_s \otimes k(h^s, v_2, \ldots, v_n) \in V^{\otimes 2} \tag{5.121}$$

is graded symmetric for any $v_2, \ldots, v_n \in V^{\otimes n-1}$.

Proof. Let $k \in HomV^{\otimes n}V$ correspond to $x = x_{i_0} \otimes \cdots \otimes x_{i_n} \in V^{\otimes n+1}$,

$$k(v_1, \ldots, v_n) := \Xi(x_{i_0} \otimes x_{i_1} \otimes x_{i_2} \otimes \cdots \otimes x_{i_n})(v_1, \ldots, v_n), \quad \text{for } v_1, \ldots, v_n \in V.$$

The symmetry of x implies that

$$x_{i_0} \otimes x_{i_1} \otimes \cdots \otimes x_{i_n} = \epsilon(\sigma) x_{i_0} \otimes x_{i_{\sigma(1)}} \otimes \cdots \otimes x_{i_{\sigma(n)}},$$

for any $\sigma \in \Sigma_n$. This means that

$$\begin{aligned} k(v_1, \ldots, v_n) &= \Xi(x_{i_0} \otimes x_{i_1} \otimes \cdots \otimes x_{i_n})(v_1, \ldots, v_n) = \\ &= \epsilon(\sigma)\Xi(x_{i_0} \otimes x_{i_{\sigma(1)}} \otimes \cdots \otimes x_{i_{\sigma(n)}})(v_1, \ldots, v_n) = (\text{by } (5.119)) \\ &= \chi(\sigma)\Xi(x_{i_0} \otimes x_{i_1} \otimes \cdots \otimes x_{i_n})(v_{\sigma^{-1}(1)}, \ldots, v_{\sigma^{-1}(n)}) = \chi(\sigma) k(v_{\sigma^{-1}(1)}, \ldots, v_{\sigma^{-1}(n)}), \end{aligned}$$

which is the antisymmetry of k.

To prove the second part of the corollary, observe that the symmetry of $x = x_0 \otimes x_{i_1} \otimes \cdots \otimes x_{i_n}$ in the first two factors,

$$x_{i_1} \otimes x_{i_0} \otimes x_{i_2} \otimes \cdots \otimes x_{i_n} = (-1)^{x_{i_0} x_{i_1}} x_{i_0} \otimes x_{i_1} \otimes x_{i_2} \otimes \cdots \otimes x_{i_n},$$

implies that

$$\begin{aligned} &\Xi(x_{i_1} \otimes x_{i_0} \otimes x_{i_2} \otimes \cdots \otimes x_{i_n})(v_1, \ldots, v_n) = \\ &\quad = (-1)^{x_{i_0} x_{i_1}} \Xi(x_{i_0} \otimes x_{i_1} \otimes x_{i_2} \otimes \cdots \otimes x_{i_n})(v_1, \ldots, v_n) \\ &\quad = (-1)^{x_{i_0}(v_1+1)} \Xi(x_{i_0} \otimes x_{i_1} \otimes x_{i_2} \otimes \cdots \otimes x_{i_n})(v_1, \ldots, v_n) \\ &\quad = (-1)^{(\deg(k)+v_1+\cdots+v_n)(v_1+1)} k(v_1, \ldots, v_n). \end{aligned}$$

Notice also that

$$(-1)^{x_{i_0}+x_{i_1}}\Xi(x_{i_0}\otimes x_{i_1}\otimes x_{i_2}\otimes\cdots\otimes x_{i_n})(h^s,v_2,\ldots,v_n)= \\ =(-1)^{\deg(k)+v_2+\cdots+v_n+1}k(h^s,v_2,\ldots,v_n),$$

because $x_{i_0}=\deg(k)+h^s+v_2+\cdots+v_n \pmod 2$ and $x_{i_1}=h^s+1 \pmod 2$ when the two sides are not zero. Thus (5.120) can be written as

$$(-1)^{(\deg(k)+v_1+\cdots+v_n)(v_1+1)}k(v_1,\ldots,v_n)= \\ = (-1)^{\deg(k)+v_2+\cdots+v_n+1}h_s\otimes B\Big(k(h^s,v_2,\ldots,v_n),v_1\Big). \tag{5.122}$$

The left-hand side of (5.122) can be obviously expressed as

$$(-1)^{(\deg(k)+v_1+\cdots+v_n)(v_1+1)}k(v_1,\ldots,v_n)= \\ = (-1)^{(\deg(k)+v_1+\cdots+v_n)h_s}k(h^s,v_2,\ldots,v_n)B(h_s,v_1). \tag{5.123}$$

Because the form B is nondegenerate, equations (5.122) and (5.123) imply that

$$(-1)^{(\deg(k)+h^s+v_2+\cdots+v_n)h_s}k(h^s,v_2,\ldots,v_n)\otimes h_s= \\ = (-1)^{\deg(k)+v_2+\cdots+v_n+1}h_s\otimes k(h^s,v_2,\ldots,v_n). \tag{5.124}$$

On the other hand, the symmetry of (5.121) means by definition that that

$$(-1)^{h^s}h_s\otimes k(h^s,v_2,\ldots,v_n)= \\ = (-1)^{h^s+(\deg(k)+h^s+v_2+\cdots+v_n)h_s}k(h^s,v_2,\ldots,v_n)\otimes h_s$$

which gives, after multiplying by the overall sign factor $(-1)^{\deg(k)+v_2+\cdots+v_n}$,

$$(-1)^{\deg(k)+h^s+v_2+\cdots+v_n}h_s\otimes k(h^s,v_2,\ldots,v_n)= \\ = (-1)^{(\deg(k)+h^s+v_2+\cdots+v_n)h^s}k(h^s,v_2,\ldots,v_n)\otimes h_s. \tag{5.125}$$

We finish the proof by observing that (5.124) is mapped to (5.125) by the endomorphism $\varphi: V\otimes V\to V\otimes V$ defined by $\varphi(u\otimes v):=(-1)^u u\otimes v$, for $u,v\in V$. ∎

Lemma 5.6.13. *For any $u_0\otimes\cdots\otimes u_k\in V^{\otimes k+1}$ and $v_0\otimes\cdots\otimes v_l\in V^{\otimes l+1}$,*

$$(-1)^{u_0+\cdots+u_{k-1}}B(u_k,v_0)\Xi(u_0\otimes u_1\otimes\cdots\otimes u_{k-1}\otimes v_1\otimes\cdots\otimes v_l)= \\ =(-1)^{l+l(u_0+\cdots+u_k)}\Big\{\Xi(u_0\otimes\cdots\otimes u_k)\circ_k\Xi(v_0\otimes\cdots\otimes v_l)\Big\}. \tag{5.126}$$

Proof. The proof is a direct verification. The left hand side of (5.126) equals, by definition, to

$$(-1)^{u_0+\cdots+u_{k-1}+(l+k-1)u_0+(l+k-2)u_1+\cdots+lu_{k-1}+(l-1)v_1+\cdots+v_{l-1}} u_0 B(u_1,-)\cdots \\ \cdots B(u_k,-)B(u_k,v_0)B(v_1,-)\cdots B(v_l,-) \tag{5.127}$$

while the right hand side of (5.126) is

$$(-1)^{l+l(u_0+\cdots+u_k)+ku_0+(k-1)u_1+\cdots u_{k-1}+lv_0+(l-1)v_0+\cdots+v_{l-1}} u_0 B(u_1,-)\cdots \\ \cdots B(u_k,-)B(u_k,v_0)B(v_1,-)\cdots B(v_l,-). \tag{5.128}$$

An easy calculation shows that the signs of (5.127) and (5.128) differ by $(-1)^{l+l(u_k+v_0)}$ which is $+1$, since $u_k+v_0=1 \pmod 2$. ■

Lemma 5.6.14. *For any* $x_0\otimes\cdots\otimes x_{n+2}\in V^{\otimes n+3}$ *and* $v_1,\ldots,v_n\in V$,

$$\begin{aligned} B(x_{n+1},x_{n+2})\Xi(x_0\otimes x_1\otimes\cdots\otimes x_n)(v_1,\ldots,v_n) &= \\ = \ (-1)^{x_{n+1}}\Xi(x_0\otimes x_1\otimes\cdots\otimes x_n\otimes x_{n+1}\otimes x_{n+2})(v_1,\ldots,v_n,h_s,h^s). \end{aligned} \tag{5.129}$$

Proof. The left hand side of (5.129) equals to

$$(-1)^{nx_0+(n-1)x_1+\cdots+x_{n-1}} x_0 B(x_1,v_1)\cdots B(x_n,v_n)B(x_{n+1},x_{n+2}), \tag{5.130}$$

while the right hand side of (5.129) is

$$(-1)^{x_{n+1}+(n+2)x_0+(n+1)x_1+\cdots+2x_n+x_{n+1}} x_0 B(x_1,v_1)\cdots \\ \cdots B(x_n,v_n)B(x_{n+1},h_s)B(x_{n+2},h^s). \tag{5.131}$$

It is immediate to see that the signs of (5.130) and (5.131) agree. The proof is finished by observing that $B(x_{n+1},h_s)B(x_{n+2},h^s)=B(x_{n+1},x_{n+2})$. ■

Proof of Theorem 5.6.10. For $n\geq 1$, $g\geq 0$ and $v_1,\ldots,v_n\in V$ let

$$k^g_n(v_1,\ldots,v_n) := \Xi(x^g_{i_0}\otimes\cdots\otimes x^g_{i_n})(v_1,\ldots,v_n).$$

Since $\deg(x^g_{i_0}\otimes\cdots\otimes x^g_{i_n})=-2(n+1)$ by assumption, $\deg(k^g_n)=n-2$.

Let us apply the map $\Xi: V^{\otimes n+1}\to \mathit{Hom}V^{\otimes n}V$ to the main equation (5.116). We claim that the result is

$$0=\sum_{\substack{k+l=n+1\\ g_1+g_2=g}}\ \sum_{\sigma\in\mathrm{unsh}(n-l,l)} (-1)^l\chi(\sigma)k^{g_1}_k(v_{\sigma(1)},\ldots,v_{\sigma(k-1)},k^{g_2}_l(v_{\sigma(k)},\ldots,v_{\sigma(n)}))(5.132) \\ + \ \frac{1}{2}(-1)^{h_s}k^{g-1}_{n+2}(v_1,\ldots,v_n,h_s,h^s).$$

Indeed, the first term of (5.116) gives

$$
\begin{aligned}
&(-1)^{x^{g_1}_{i_k}} B(x^{g_1}_{i_k}, x^{g_2}_{j_0})\Xi\Big(x^{g_1}_{i_0}\,\mathrm{Sh}(x^{g_1}_{i_1}\otimes\cdots\otimes x^{g_1}_{i_{k-1}}|x^{g_2}_{j_1}\otimes\cdots\otimes x^{g_2}_{j_l})\Big)(v_1,\ldots,v_n) = (\text{by } (5.119))\\
&= \sum \chi(\sigma)(-1)^{x^{g_1}_{i_k}} B(x^{g_1}_{i_k}, x^{g_2}_{j_0})\Xi(x^{g_1}_{i_0}\otimes\cdots\otimes x^{g_1}_{i_{k-1}}\otimes x^{g_2}_{j_1}\otimes\cdots\otimes x^{g_2}_{j_l})(v_{\sigma(1)},\ldots,v_{\sigma(n)}) =\\
&= (\text{by } (5.126)) \quad \sum \chi(\sigma)(-1)^l\Big\{\Xi(x^{g_1}_{i_0}\otimes\cdots\otimes x^{g_1}_{i_{k-1}})\circ_k \Xi(x^{g_2}_{j_1}\otimes\cdots\otimes x^{g_2}_{j_l})\Big\}(v_{\sigma(1)},\ldots,v_{\sigma(n)})\\
&= \sum \chi(\sigma)(-1)^l(k^{g_1}_k\circ_k k^{g_2}_l)(v_{\sigma(1)},\ldots,v_{\sigma(n)}),
\end{aligned}
$$

which is the first term of (5.132). For brevity, we did not specify the ranges of summations which are the same as in (5.132). We also used the identity $x^{g_1}_{i_0}+\cdots+x^{g_1}_{i_k} = 0 \pmod 2$. The second term of (5.116) gives

$$
\begin{aligned}
&-\frac{1}{2}B(x^{g-1}_{i_{n+1}}, x^{g-1}_{i_{n+2}})\Xi(x^{g-1}_{i_0}\otimes\cdots\otimes x^{g-1}_{i_n})(v_1,\ldots,v_n) = (\text{by } (5.129))\\
&= -\frac{1}{2}(-1)^{x^{g-1}_{i_{n+1}}}\Xi(x^{g-1}_{i_0}\otimes\cdots\otimes x^{g-1}_{i_{n+2}})(v_1,\ldots,v_n,h_s,h^s) =\\
&= \frac{1}{2}(-1)^{h_s}k^{g-1}_{n+2}(v_1,\ldots,v_n,h_s,h^s),
\end{aligned}
$$

since $x^{g-1}_{i_{n+1}} = h_s+1 \pmod 2$. The operations k^g_n are, by Corollary 5.6.12, antisymmetric, so we can rewrite (5.132) as

$$
\begin{aligned}
0 = \sum (-1)^{k(l+1)+l(v_{\sigma(l+1)}+\cdots+v_{\sigma(n)})}\chi(\sigma)k^{g_1}_k(k^{g_2}_l(v_{\sigma(1)},\ldots,v_{\sigma(l)}),v_{\sigma(l+1)},\ldots,v_{\sigma(n)})-\\
-\ \frac{1}{2}(-1)^{h_s+v_1+\cdots+v_n}k^{g-1}_{n+2}(h_s,h^s,v_1,\ldots,v_n)
\end{aligned}
$$

where the summation is the same as in (5.132). The substitution

$$
\overline{k}^g_n(v_1,\ldots,v_n) := (-1)^{n(v_1+\cdots+v_n)}k^g_n(v_1,\ldots,v_n)
$$

converts the above equation to

$$
\begin{aligned}
0 = \sum (-1)^{k+(n+1)(v_1+\cdots+v_n)}\chi(\sigma)\overline{k}^{g_1}_k(\overline{k}^{g_2}_l(v_{\sigma(1)},\ldots,v_{\sigma(l)}),v_{\sigma(l+1)},\ldots,v_{\sigma(n)})-\\
-\ \frac{1}{2}(-1)^{h_s+(n+1)(v_1+\cdots+v_n)+n}\overline{k}^{g-1}_{n+2}(h_s,h^s,v_1,\ldots,v_n).
\end{aligned}
$$

Multiplying by the overall sign $(-1)^{(n+1)(v_1+\cdots+v_n)}$ it can be further simplified to

$$
\begin{aligned}
0 = \sum (-1)^k\chi(\sigma)\overline{k}^{g_1}_k(\overline{k}^{g_2}_l(v_{\sigma(1)},\ldots,v_{\sigma(l)}),v_{\sigma(l+1)},\ldots,v_{\sigma(n)})-\\
-\ \frac{1}{2}(-1)^{h_s+n}\overline{k}^{g-1}_{n+2}(h_s,h^s,v_1,\ldots,v_n).
\end{aligned}
\tag{5.133}
$$

Finally, the substitution

$$
\begin{aligned}
l^g_n(v_1,\ldots,v_n) &:= (-1)^{\frac{n(n+1)}{2}}\overline{k}^g_n(v_1,\ldots,v_n) =\\
&= (-1)^{\frac{n(n+1)}{2}+n(v_1+\cdots+v_n)}k^g_n(v_1,\ldots,v_n)
\end{aligned}
\tag{5.134}
$$

converts (5.133) to

$$\sum(-1)^{l(k-1)}\chi(\sigma)l_k^{g_1}(l_l^{g_2}(v_{\sigma(1)},\ldots,v_{\sigma(l)}),v_{\sigma(l+1)},\ldots,v_{\sigma(n)})+ \\ + \frac{1}{2}(-1)^{h_s+n}l_{n+2}^{g-1}(h_s,h^s,v_1,\ldots,v_n),$$

which is, quite miraculously, (5.101). It remains to show that the element

$$(-1)^{(n+1)h_s}h_s\otimes l_n^g(h^s,v_2,\ldots,v_n)\in V\otimes V,$$

where l_n^g is given by (5.134), is symmetric. In terms of the operations k_n^g it is the same as

$$(-1)^{(n+1)h_s+\frac{n(n+1)}{2}+n(h^s+v_2+\cdots+v_n)}h_s\otimes k_n^g(h^s,v_2,\ldots,v_n)= \\ = (-1)^{h_s+\frac{n(n+1)}{2}+n(1+v_2+\cdots+v_n)}h_s\otimes k_n^g(h^s,v_2,\ldots,v_n). \tag{5.135}$$

On the other hand, the element

$$(-1)^{h_s}h_s\otimes k_n^g(h^s,v_2,\ldots,v_n)$$

whose symmetry is guaranteed by Corollary 5.6.12, differs from (5.135) only by an overall sign factor $(-1)^{\frac{n(n+1)}{2}+n(1+v_2+\cdots+v_n)}$, thus (5.135) is symmetric, too. ■

Contents

Glossary of notations.

Aff	group of affine transformations of $\mathbb{C}$, 177
$\overline{\mathcal{M}} = \{\overline{\mathcal{M}}(n)\}_{n\geq 2}$	configuration pseudo-operad, 180
$\overline{\mathcal{M}}(n)$	compactification of the moduli space $\mathcal{M}_{0,n+1}$, 176
$\overline{\mathsf{Con}}(M,n)$	compactification of the configuration space $\mathsf{F}(M,n)$, 198
$\mathsf{f}\overline{\mathsf{Con}}(Mn)$	compactification of the framed configuration space $f\mathsf{F}(M,n)$, 198
bld	blowing-down map, 204
$\mathcal{C}mon$	operad for commutative associative topological monoids, 85
$\mathsf{F}(M,n)$	configuration space of n distinct labelled points in a manifold M, 198
$\mathrm{edg}(T)$	set of internal edges of a tree T, 181
$f\mathsf{F}(M,n)$	framed configuration space of n distinct labelled points in a manifold M, 198
$\mathrm{Met}(T)$	space of all metrics on the set of edges of a tree T, 184
$\mathcal{M}_{0,n+1}$	moduli space of n distinct numbered points on the complex projective line, 176
$\mathcal{M}on$	operad for associative topological monoids, 85
$\mathrm{rv}(T)$	vertex adjacent to the root of a rooted tree T, 198
$\mathcal{T}m(n)$	space of metric n-trees, 83
$\mathtt{Tree}^e_n$	set of all n-trees with all vertices except possibly the vertex adjacent to the root at least binary, 198
$W\mathcal{O}$	W-construction on the operad $\mathcal{O}$, 82

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References.

[1] S. Axelrod and I.M. Singer. Chern-Simons perturbation theory II. *Jour. Diff. Geom.*, 39:173–213, 1994.

[2] J. A. Azcárraga and J. C. Pérez Bueno. Higher-order simple Lie algebras. *Comm. Math. Phys.*, 184:669–681, 1997.

[3] D. Balavoine. Homology and cohomology with coefficients, of an algebra over a quadratic operad. *Jour. of Pure and Appl. Algebra*, 132:221–258, 1998.

[4] M. Barr. Cohomology in tensored categories. In M. Kelly and S. Eilenberg, editors, *Conference on Categorical Algebra*, pages 344–354. Springer-Verlag, Berlin, Heidelberg, New York, 1966.

[5] M. Baues, H.-J. Jibladze and A. Tonks. Cohomology of monoids in monoidal categories. *Contemp. Mathematics*, 202:137–165, 1997.

[6] A. Beilinson and V. Ginzburg. Infinitesimal structure of moduli spaces of g-bundles. *Internat. Math. Res. Notices*, pages 63–74, 1992.

[7] A. Beilinson, V. Ginzburg, and V. Schechtman. Koszul duality. *J. Geom. Phys.*, 5:317–350, 1988.

[8] R.L. Bishop and R.J. Crittenden. *Geometry of Manifolds*, volume XV of *Applied Mathematics*. Academic Press, New York-London, 1964.

[9] J.M. Boardman and R.M. Vogt. Homotopy-everything H-spaces. *Bull. Amer. Math. Soc.*, 74(6):1117–1122, 1968.

[10] J.M. Boardman and R.M. Vogt. *Homotopy Invariant Algebraic Structures on Topological Spaces*. Springer-Verlag, 1973.

[11] R. Bott and C. Taubes. On the self-linking of knots. *Journal of Math. Physics*, 35(10):5247–5287, October 1994.

[12] N. Bourbaki. Algèbre. Chapitre X: Algèbre Homologique. Mason, 1980.

[13] W. Browder. Homology operations and loop spaces. *Illinois Jour. of Math.*, 4(3):347–357, 1960.

[14] P. Cartier. Homologie cycliques: rapport sur des travaux récents de Connes, Karoubi, Loday, Quillen ... Séminaire Bourbaki 621, 1983–84.

[15] C. Chevalley. Review of Lois des groupes et analyseurs. In *Math. Reviews*, volume 17. American Mathematical Society, 1956.

[16] F.R. Cohen. The homology of $\mathcal{C}_{n+1}$-spaces, $n \geq 0$. In *The homology of iterated loop spaces*, volume 533 of *Lecture Notes in Math.* Springer-Verlag, 1976.

[17] F.R. Cohen. Artin's braid groups, classical homotopy theory, and sundry other curiosities. *Contemp. Math.*, 78:167–206, 1988.

[18] A. Connes. Non commutative differential geometry. Prépublications I.H.E.S., March 1983.

[19] M. Culler and K. Vogtmann. Moduli of graphs and automorphisms of free groups. *Inventiones Math.*, 84:91–119, 1986.

[20] P. Deligne. Théorie de Hodge II. *Inst. Hautes Études Sci. Publ. Math.*, 40:5–58, 1971.

[21] P. Deligne. Resumé des premiérs exposés de A. Grothendieck. In *Groupes de Monodromie en Déometrie Algébrique*, volume 288 of *Lecture Notes in Math.*, pages 1–24. Springer-Verlag, Berlin, 1972.

[22] P. Deligne. A letter to Stasheff, Gerstenhaber, May, Schechtman and Drinfeld. Unpublished, 1993.

[23] P. Deligne and D. Mumford. The irreducibility of the space of curves of given genus. *Publ. I.H.E.S.*, 36(75), 1969.

[24] E. Dyer and R.K. Lashof. Homology of iterated loop spaces. *American Journal of Mathematics*, 84(1):35–88, 1962.

[25] S. Eilenberg and J.C. Moore. Adjoint functors and triples. *Illinois Jour. Math.*, 9:381–389, 1965.

[26] H. Federer. *Geometric Measure Theory.* Springer-Verlag, Berlin-Heidelberg-New York, 1969.

[27] Y. Félix, S. Halperin, and J-C. Thomas. Differential graded algebras in topology. In I.M. James, editor, *Handbook of algebraic topology*, pages 829–865. North-Holland, Amsterdam, 1995.

[28] K. Fukaya. Floer homology, A^∞-categories and topological field theory. In J. Andersen, J. Dupont, H. Pertersen, and A. Swan, editors, *Geometry and Physics*, volume 184 of *Lecture Notes in Pure and Applied Mathematics*, pages 9–32. Marcel-Dekker, 1995. Notes by P. Seidel.

[29] W. Fulton and R. MacPherson. A compactification of configuration spaces. *Annals of Mathematics*, 139:183–225, 1994.

[30] Bredon G.E. *Sheaf Theory.* McGraw-Hill, New York, 1967.

[31] M. Gerstenhaber. The cohomology structure of an associative ring. *Annals of Mathematics*, 78(2):267–288, 1963.

[32] M. Gerstenhaber. On the deformation of rings and algebras. *Annals of Mathematics*, 79,1:59–104, 1964.

[33] M. Gerstenhaber and A.A. Voronov. Vyschie operacii na komplekse Hochschilda. *Fukcion. Analiz i ego Pril.*, 29(1):1–6, 1995. In Russian.

[34] E. Getzler. Batalin-Vilkovisky algebras and two-dimensional topological field theories. *Commun. Math. Phys.*, 159:265–285, 1994.

[35] E. Getzler. Two-dimensional topological gravity and equivariant cohomology. *Comm. Math. Phys.*, 163(3):473–489, 1994.

[36] E. Getzler and J.D.S. Jones. Operads, homotopy algebra, and iterated integrals for double loop spaces. Preprint `hep-th/9403055`, 1993.

[37] E. Getzler and M.M. Kapranov. Cyclic operads and cyclic homology. In S.-T. Yau, editor, *Geometry, Topology and Physics for Raoul Bott*, volume 4 of *Conf. Proc. Lect. Notes. Geom. Topol.*, pages 167–201. International Press, 1995.

[38] E. Getzler and M.M. Kapranov. Modular operads. *Compositio Math.*, 110(1):65–126, 1998.

[39] V. Ginzburg. Resolution of diagonals and moduli space. In R. Dijkgraaf, C. Faber, and G. van der Geer, editors, *The Moduli Space of Curves*, volume 129 of *Progress in Mathematics*, pages 231–266. Birkhäuser, 1995.

[40] V. Ginzburg and M.M. Kapranov. Koszul duality for operads. *Duke Math. Journal*, 76(1):203–272, 1994.

[41] A.V. Gnedbaye. Opérades des algébres $(k+1)$-aires. In J. Loday, J.-L. Stasheff and A.A. Voronov, editors, *Operads: Proceedings of Renaissance Conferences*, volume 202 of *Contemporary Mathematics*, pages 83–114. Amer. Math. Soc., 1996.

[42] A. Grothendieck. Produits tensoriels et produits topologiques. *Mem. Amer. Math. Soc.*, 16, 1955.

[43] P. Hanlon and M.L. Wachs. On Lie k-algebras. *Advances in Math.*, 113:206–236, 1995.

[44] J Harer. The cohomology of the moduli space of curves. In *Theory of Moduli*, number 1337 in Lecture Notes in Mathematics, pages 138–221. Springer-Verlag, 1988.

[45] R. Hartshorne. *Algebraic Geometry.* Number 52 in Graduate Texts in Mathematics. Springer-Verlag, 1977.

[46] M. Herrera and D. Lieberman. Residues and principal values on complex spaces. *Math. Annalen*, 194:259–294, 1971.

[47] P.J. Hilton and U. Stammbach. *A Course in Homological Algebra*, volume 4 of *Graduate texts in Mathematics.* Springer-Verlag, 1971.

[48] V. Hinich. Homological algebra of homotopy algebras. *Comm. in Algebra*, 10(25):3291–3323, 1997.

[49] V.A. Hinich and V.V. Schechtman. On homotopy limit of homotopy algebras. *Lecture Notes in Mathematics*, 1289:240–264, 1987.

[50] V.A. Hinich and V.V. Schechtman. Homotopy Lie algebras. *Advances in Soviet Mathem.*, 16(2):1–28, 1993.

[51] Y.-Z. Huang. Operadic formulation of topological vertex algebras and Gerstenhaber or Batalin-Vilkovisky algebras. *Commun. Math. Phys.*, 164:145–161, 1994.

[52] Y.-Z. Huang and J. Lepowsky. Vertex operator algebras and operads. In *The Gelfand Mathematics Seminars, 1990–1992*, pages 145–161. Birkhäuser, Boston, 1993.

[53] H. Jarchov. *Locally Convex Spaces.* Teubner, Stuttgart, 1981.

[54] A. Joyal. Une théorie combinatoire des séries formelle. *Advances in Math.*, 42:1–82, 1981.

[55] M. Kapranov. The permutoassociahedron, Mac Lane's coherence theorem and asymptotic zones for the KZ equation. *Journal of Pure and Appl. Algebra*, 85:119–142, 1993.

[56] J.L. Kelley. *General Topology*. D. Van Nostrand, 1957.

[57] G.M. Kelly. On Mac Lane's conditions for coherence of natural associativities, commutativities, etc. *Jour. of Algebra*, 1:397–402, 1964.

[58] T. Kimura, J.D. Stasheff, and A.A. Voronov. On operad structures of moduli spaces and string theory. *Comm. Math. Phys.*, 171:1–25, 1995.

[59] F.F. Knudsen. The projectivity of the moduli space of stable curves, II: The stacks $M_{g,n}$. *Math. Scand.*, 52:161–199, 1983.

[60] F.F. Knudsen and D. Mumford. The projectivity of the moduli space of stable curves I: Preliminaries on "det" and "Div". *Math. Scand.*, 39:19–55, 1976.

[61] M. Kontsevich. Feynman diagrams and low-dimensional topology. Preprint, 1991.

[62] M. Kontsevich. Formal (non)commutative symplectic geometry. In *The Gel'fand mathematics seminars 1990–1992*. Birkhäuser, 1993.

[63] M. Kontsevich. Feynman diagrams and low-dimensional topology. In *First European Congress of Mathematics II*, volume 120 of *Progr. Math.* Birkhäuser, Basel, 1994.

[64] M. Kontsevich and Yu. Manin. Gromov-Witten classes, quantum cohomology, and enumerative geometry. Preprint.

[65] M. Kontsevich and Y. Soibelman. Deformations of algebras over operads and Deligne's conjecture. Preprint `math.QA/0001151`, January 2000.

[66] I. Kriz and J.P. May. *Operads, Algebras, Modules and Motives*, volume 233 of *Astérisque*. Société Mathématique de France, 1995.

[67] I. Kř´I ž and J.P. May. Operads, algebras, and modules, I: Definitions and examples. Preprint, University of Chicago, 1993.

[68] F.W. Lawvere. Functorial semantics of algebraic theories. *Proc. Nat. Acad. Sci. USA*, 50:869–872, 1963.

[69] M. Lazard. Lois des groupes et analyseurs. *Ann. Sci. Ecole Norm. Sup.*, 72(3):299–400, 1955.

[70] M. Lazard. Lois des groupes et analyseurs. In *Séminaire Bourbaki, 7e année 1954/1955*, volume 109, pages 1–15. Soc. Math. Francé, 1959.

[71] C. Lee. Some notes on triangulating polytopes. In *3. Kolloquium über Diskrete Geometrie*, pages 173–181. Institut für Mathematik, Universität Salzburg, 1985.

[72] C. Lee. The associahedron and triangulations of the n-gon. *Europ. Jour. Combinatorics*, 10:551–560, 1989.

[73] D. Lehmann. *Théorie Homotopique des Forms Différentielles*, volume 54. Soc. Math. France, 1977.

[74] J.-L. Loday and D. Quillen. Cyclic homology and the Lie algebra homology of matrices. *Comment. Math. Helv.*, 59:565–591, 1984.

[75] S. Mac Lane. Natural associativity and commutativity. *Rice Univ. Studies*, 49.1:28–46, 1963.

[76] S. Mac Lane. Categorical algebra. *Bull. Amer. Math. Soc.*, 71:40–106, 1965.

[77] P. Malraison. Homotopy associative categories. In *Lecture Notes in Math.*, volume 428, pages 108–131. Springer-Verlag, 1974.

[78] M.A. Mandell. E_∞-algebras and p-adic homotopy theory. Preprint, 1998.

[79] M. Markl. Loop homotopy algebras in closed string field theory. Preprint hep-th/9711045.

[80] M. Markl. Cotangent cohomology of a category and deformations. *Journal of Pure and Appl. Algebra*, 113:195–218, 1996.

[81] M. Markl. Models for operads. *Communications in Algebra*, 24(4):1471–1500, 1996.

[82] M. Markl. Simplex, associahedron, and cyclohedron. In J. McCleary, editor, *Higher Homotopy Structures in Topology and Mathematical Physics*, volume 227 of *Contemporary Mathematics*, pages 235–265. Amer. Math. Soc., 1998.

[83] M. Markl. A compactification of the real configuration space as an operadic completion. *Journal of Algebra*, 215:185–204, 1999.

[84] M. Markl. Cyclic operads and homology of graph complexes. *Supplem. ai Rend. Circ. Matem. Palermo*, 59:161–170, 1999.

[85] M. Markl. Homotopy algebras are homotopy algebras. Preprint `math.AT/9907138`, July 1999.

[86] M. Markl. Homotopy algebras via resolutions of operads. *Supplem. ai Rend. Circ. Matem. Palermo, Ser. II*, 63:157–164, 2000.

[87] M. Markl. Ideal perturbation lemma. Preprint `math.AT/0002130`, February 2000.

[88] M. Markl and S. Shnider. Coherence without commutative diagrams, Lie-hedra and other curiosities. Preprint q-alg/9712027, December 1997.

[89] J.P. May. A general algebraic approach to Steenrod operations. In F.P. Peterson, editor, *The Steenrod Algebra and its Applications*, volume 168 of *Lectures Notes on Mathematics*, pages 153–231. Springer Verlag, 1970.

[90] J.P. May. *The Geometry of Iterated Loop Spaces*, volume 271 of *Lecture Notes in Mathematics*. Springer-Verlag, 1972.

[91] J.P. May. Brave new worlds in stable homotopy theory. In M. Mahowald, editor, *Proceedings of a Conference on Homotopy Theory, Evanston, IL, USA*, volume 220 of *Contemp. Math.*, pages 193–212. American Mathematical Society, 1998.

[92] J.P. May et al. *Equivariant Homotopy and Cohomology Theory.* American Mathematical Society, 1996.

[93] J.C. Moore. Differential homological algebra. *Actes Congres Intern. Math.*, 1:335–339, 1971.

[94] R.E. Mosher and M.C. Tangora. *Cohomology Operations and Applications in Homotopy Theory.* Harper & Row, New York, Avanston and London, 1968.

[95] M. Penkava. Infinity algebras and the cohomology of graph complexes. Preprint, 1995.

[96] M. Penkava and A. Schwarz. A_∞ algebras and the cohomology of moduli spaces. *Trans. Amer. Math. Soc.*, 169:91–107, 1995.

[97] R.C. Penner. The decorated Teichmüller space of punctured surfaces. *Commun. Math. Phys.*, 113:299–339, 1987.

[98] T. Pirashvili. On Leibniz homology. *Annales de l'Institut Fourier*, 44(2):401–411, 1994.

[99] S.B. Priddy. Koszul resolutions. *Trans. Amer. Math. Society*, 152:39–60, 1970.

[100] D. Quillen. Homotopical algebra. *Lecture Notes in Math.*, 43, 1967.

[101] D. Quillen. Rational homotopy theory. *Annals of Math.*, 90:205–295, 1969.

[102] D. Quillen. Cohomology of groups. *Actes Congrès International Math.*, 2:47–51, 1970.

[103] R. Ree. Lie elements and an algebra associated with shuffles. *Annals of Math.*, 68:210–220, 1958.

[104] V. S. Retakh. Lie-Massey brackets and n-homotopy multiplicative maps of differential graded Lie algebras. *Journal of Pure and Appl. Algebra*, 89:217–229, 1993.

[105] D. Sahoo and M.C. Valsakumar. Nambu mechanics and its quantization. *Phys. Rev. A*, 46:4410–4412, 1992.

[106] S. Saneblidze and R. Umble. A diagonal on the associahedra. Preprint, March 2000.

[107] M. Schlessinger and J.D. Stasheff. Deformation theory and rational homotopy type. Preprint, 1984.

[108] M. Schlessinger and J.D. Stasheff. The Lie algebra structure of tangent cohomology and deformation theory. *Jour. Pure Appl. Algebra*, 38:313–322, 1985.

[109] S. Shnider and S. Sternberg. *Quantum groups: from coalgebras to Drinfeld algebras.* Graduate texts in mathematical physics. International Press, 1994.

[110] V.A. Smirnov. Gomologii B-konstrukcij i ko-B-konstrukcij. Preprint, in Russian.

[111] V.A. Smirnov. Homotopy theory of coalgebras. *Izv. Akad. Nauk. SSSR, Ser. Mat.*, 49(6):1302–1321, 1985. In Russian.

[112] V.A. Smirnov. Secondary operations in the homology of the operad e. *Izv. RAN*, 56:449–468, 1992.

[113] J.R. Smith. *Iterating the Cobar Construction*, volume 109 of *Memoirs of the AMS*. American Mathematical Society, 1994.

[114] E.H. Spanier. *Algebraic Topology.* McGraw-Hill, New York, 1966.

[115] J. Stasheff. An operad-chik looks at configuration spaces, moduli spaces and mathematical physics. In J. Loday, J.-L. Stasheff and A.A. Voronov, editors, *Operads: Proceedings of Renaissance Conferences*, volume 202 of *Contemporary Mathematics*, pages 53–82. Amer. Math. Soc., 1996.

[116] J.D. Stasheff. On the homotopy associativity of H-spaces, I. Dissertation, Princeton University, 1961.

[117] J.D. Stasheff. Homotopy associativity of H-spaces I,II. *Trans. Amer. Math. Soc.*, 108:275–312, 1963.

[118] J.D. Stasheff. From operads to 'physically' inspired theories. In J.D. Stasheff J.-L. Loday and A.A. Voronov, editors, *Operads: Proceedings of Renaissance Conferences*, volume 202 of *Contemp. Mathematics*, pages 53–81, 1997.

[119] N.E. Steenrod and D.B.A. Epstein. *Cohomology Operations*. Princeton University Press, 1962.

[120] M. Sugawara. On a condition that a space is an h-space. *Math. J. Okayama Univ.*, 6:109–129, 1957.

[121] M. Sugawara. On a condition that a space is group-like. *Math. J. Okayama Univ.*, 7:123–149, 1957.

[122] M. Sugawara. On the homotopy-commutativity of groups and loop spaces. *Mem. Coll. Sci. Univ. Kyoto, Ser. A Math.*, 33:257–269, 1960/61.

[123] D. Sullivan. Infinitesimal computations in topology. *Publ. IHES*, 47:269–331, 1977.

[124] L. Takhtajan. On foundation of the generalized Nambu mechanics. *Comm. Math. Phys.*, 160:295–315, 1994.

[125] D.E. Tamarkin and B. Tsygan. Noncommutative differential calculus, homotopy BV algebras and formality conjectures. Preprint `math.KT/0002116`, February 2000.

[126] D. Tanré. *Homotopie Rationnelle: Modèles de Chen, Quillen, Sullivan.* Lect. Notes in Math. 1025. Springer-Verlag, 1983.

[127] B.L. Tsygan. Homology of matrix algebras over rings and Hochschild homology. *Uspekhi Mat. Nauk*, 38:217–218, 1983. In Russian.

[128] A. Vinogradov and M. Vinogradov. On multiple generalizations of Lie algebras and Poisson manifolds. In *Secondary calculus and cohomological physics, Moscow, 1977.* Amer. Math. Soc., 1998.

[129] A. Vistoli. Intersection theory on algebraic stacks and on their moduli spaces. *Invent. Math.*, 97:613–670, 1989.

[130] R.M. Vogt. My time as Mike Boardman's student and our work on infinite loop spaces. Preprint, 1998.

[131] R.M. Vogt. Cofibrant operads and universal E_∞ operads. In R. Vogt, editor, *Workshop on Operads, Osnabrück, June 1998*, Preprint of Univ. Bielefeld, pages 81–89, 1999.

[132] A. A. Voronov. Homotopy Gerstenhaber algebras. Preprint `math.QA/9908040`, August 1999.

[133] C. Weibel. e-letter to Stasheff.

[134] R.O. Wells. *Differential Analysis on Complex Manifolds*, volume 65 of *Graduate Texts in Mathematics*. Springer-Verlag, 1980.

[135] A.N. Whitehead. *A Treatise on Universal Algebra.* Cambridge, 1898.

[136] B. Zwiebach. Closed string field theory: Quantum action and the Batalin-Vilkovisky master equation. *Nuclear Physics*, B390:33–152, 1993.